FINAL REPORT

Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

work done

on the

UGC Major Research Project

No. 42 – 39/2013(SR)

Bhatt Milind Bhanuprasad
Department of Statistics
Sardar Patel University
Vallabh Vidya Nagar

Final Report of the work done on the Major Research Project No. 42-39/2013(SR)

INDEX

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

Final Report of the work done on the Major Research Project No. 42-39/2013(SR)

INDEX

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

FINAL Report of the work done on the Major Research Project No. 42-39/2013(SR)

FINAL REPORT : MY CONTRIBUTION TO SOCIETY

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

(1) SUMMARY OF RESEARCH

Knowing characterizing property which may provide unexpectedly accurate information about distributions and one can recognize a class of distributions before any statistical inference is made. This feature of characterizations of probability distributions is peculiar to characterizing property and attracted attention of both theoretician and applied workers but there is no general theory of it. Various approaches are available in the literature. Most of them are research concentrated on the role of first two moments.

Characterizations was independently develop in different branches of applied probability and pure mathematics. Characterizations theorem are located on borderline between probability theory and mathematical statistics. It is of general interest to mathematical community, to probabilists and statistician as well as to researchers and practitioner industrial engineering and operation research and various scientist specializing in natural and behavior science, in particular those who are interested in foundation and application of probabilistic model building. (see basic book on characterizations by [1]Lukacs and Laha and the more advance comprehensive mathematical tools (entirely toward normal distribution) see [2] kagan, Linnik and Rao.

It is well known that smaller and the larger of a random sample of size two are positively correlate and coefficient of correlation is less or equal to one half. [3] Bartoszyn'ski proposed that a result of this type might exist in connection with a problem in cell division. Since the two daughter cells cannot always be distinguished later, the times till their further division can only be recorded as the earlier event and the later event. The correlation between these ordered pairs thus may provide the only information on the independence of the two events. Path breaking different approach

Normally the mass of a root has a uniform distribution. Plant develops into the reproductive phase of growth, a mat of smaller roots grows near the surface to a depth of approximately 1/6-th of maximum depth achieve [See G. Ooms and K. L. Moore [4]]. Dixit [5] studied problem of efficient estimation of parameters of a uniform distribution in the presence of outliers and these masses of have different masses therefore, those masses have different uniform distribution with unknown parameters and distributed with Generalize Uniform Distribution (GUD).

(2) MY CONTRIBUTION

Provide new path braking unified approach "identity of distribution and equality of expectation of function of random variable, function of order statistics" to characterize Negative Exponential Distribution, Power-Function Distribution, Pareto Distribution, Uniform Distribution and generalize Uniform Distribution through expectation and through expectation of function of order statistics, over and above other approaches available in statistics litterer such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy, independence of suitable function of order statistics, linear relation of conditional expectation, recurrence relations between expectations of function of order statistics, distributional properties of exponential distributional, record valves, lower record statistics, product of order statistics and Lorenz curve, etc.

REFERENCE

1 Lukacs, E and Laha, R. G. (1964). Application of characteristic function. Griffin, London.

2 Kagan, A.M., Linnik, Yu.V. and Rao, C.R. (1973). *Characterization Problems in Mathematical Statistics,* J. Wiley, New York (English translation).

3 Bartoszynski, R. (1980). Personal communication.

4 G. Ooms, and K. L. Moore. A model assay for genetic and environmental changes in the architecture of intact roots systems of plants, grown in vitro, Plant Cell, Tissue and Organ Culture, 27, 129-139, 1991.

5 U. J. Dixit and Masoom, Ali and J. Woo. Efficient estimation of parameters of a uniform distribution in the presence of outliers, Soochow Journal of mathematics, Volume 29, No. 4, pp. 363-369, October 2003.

(3) Presented research paper :

SR. NO.	Title of the paper presented	Title of conference/ Seminar	Organized by
1	Characterization of Power-Function Distribution through Expectation	3rd International Science Congress (ISC-2013)	**8th & 9th December, 2013** International Science Congress Association Under the auspices of Karunya University, Coimbatore, Tamil Nadu, INDIA
	Characterization of Negative Exponential	International conference on role of statistics in the	**December 16-18, 2013.** Department of statistics

FINAL Report of the work done on the Major Research Project No. 42-39/2013(SR)

FINAL REPORT : MY CONTRIBUTION TO SOCIETY

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

	Distribution through Expectation	advancement of science and technology, on the occasion of diamond jubilee year of the Department of statistics university of pune & the international year of statistics,	university of pune
2	Characterization of Pareto Distribution through Expectation Of function of Order Statistics	International Conference on Operations Research for Data Analytics and Decision Analysis (ICORDADA-13)	**October 21-23,2013** Department of Statistics, University of Kashmir, Srinagar, J&K, India (NAAC Accredited Grade "A")
	Characterization of Negative Exponential Distribution through Expectation	International Conference on Recent Advances in Statistics and Their Applications in conjunction with xxxiii annual convention of Indian society for probability and statistics (ISPS)	**26th-28th December, 2013** Dept. of Statistics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431004, State: Maharashtra (INDIA)
3	Characterization of Uniform Distribution u (0,θ) through Expectation	1st international Virtual Congress	**2014 Jun 5-10** Organized by International Science Congress Association, www.isca.net.co
4	Characterization of Generalized Uniform Distribution U(0,θ) Through Expectation Of function of Order Statistics	32nd Annual National Conference on Recent Advances in Statistical Methods And Their Applications in Health Sciences	**November 1-3, 2014** ISMSCON 2014 Department of Statistics, University of Jammu,
5	Characterization of Generalized Uniform Distribution U(0,θ) Through Expectation	International Conference on Statistics and Information Technology for a Growing Nation	**2014, Nov. 30 - Dec. 2** ISPS Department of Statistics, Sri Venkateswara University, Tirupati.
6	Characterization of Power-	International conference on	**2014,** Dec. 1- **3**

FINAL Report of the work done on the Major Research Project No. 42-39/2013(SR)

FINAL REPORT : MY CONTRIBUTION TO SOCIETY

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

	Function Distribution through Expectation	Role of Statistics in International Conference on Operations Research or for Big Data and Analytics	ORSI Department of Statistics, Sri Venkateswara University, Tirupati
6	Characterization through Expectation	International Workshop on Reliability and Time Series Methodology relevant to Business and Industry	**2015 Jan. 5 - 7** Department. of Statistics Cochin Univ. of Science and Technology, Keral
7	Statistical Skill	1st International Young Scientist Congress (IYSC-2015) for Workshop on Statistical Skill	**8th & 9th August, 2013** ISCA Maharaja Ranjit Singh College of Sciences, Indore,
8	Characterization of one-truncation parameter family of distributions through Expectation	5th International Science Congress (ISC-2015) Conference Divers Resources: Solutions and Advancements	**2015 Dec. 8 - 9** ISCA Tribhuvan University, Kathmandu, Nepal
9	characterization of Pareto distribution through expectation of function of order statistics	International Conference OR best practices & operational issues in development sector	**2015 Dec.17 - 19** ORSI-Bhubaneswar Chapter & SOA University
10	characterization of one truncation parameter families of distributions through expectation of function of order statistics	International Conference on Celebrating Statistical Innovation and Impact in a World of Big & Small Data	**2015 Dec.20 - 24** IISA International Indian Statistical Association Department of Statistics Savitribai Phule Pune University.
11	Deliver three lectures		**21-25/6-2016** in Department of Statistics University of Kashmi
12	characterizing non-regular parameter families of distributions	International Workshop on Reliability Theory and Survival Analysis	**2016 Nov. 3 – 4** Department of Statistics, Savitribai Phule Pune University,

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

13	Non-regular (truncation parameter) families	6th International Science Congress	**2016 Dec. 8 – 9** ISCA Hutatma Rajguru Mahavidyalaya, Pune,
14	Participated in workshop conducted on Data Envelopment Analysis	Data Envelopment Analysis	2016 Dec. 11 Birla Institute of Management Technology, Greater Noida
15	characterization of power function distribution through expectation of function of order statistics	International Conference on Analytics in Operational Research	**2016 Dec.12 - 14** ORSI Birla e of Management Technology

(4) Published 6 research paper (attached) in international journals

SR. NO.	Title of the published paper
1	Bhatt Milind B. (2013), Characterization of Negative Exponential Distribution through Expectation, Open Journal of Statistics, 2013, 3, 367-369.
2	Bhatt Milind B. (2013), Characterization of Power-Function Distribution through Expectation, Open Journal of Statistics, 2013, 3, 441-443.
3	Bhatt Milind B. (2014), Characterization of Negative Exponential Distribution Through Expectation of Function of Order Statistics, Journal of Statistical Science and Application 2 (2014) 79-84.
4	Bhatt Milind B. (2014), Characterization of Uniform Distribution u (0,) through Expectation, Research Journal of Mathematical and Statistical Sciences, (2014) February Vol. 2(2), 16-19.
5	Bhatt Milind B. (2014), Characterization of Uniform Distribution through Expectation of Function of Order Statistics, Research Journal of Mathematical and Statistical Sciences, August (2014), Vol. 2(8), 10-14.
6	Bhatt Milind B. (2014), Characterization of Generalized Uniform Distribution through Expectation, Open Journal of Statistics, 2014, 4, 563-569.
7	Bhatt Milind B. (2014), Characterization of Generalized Uniform Distribution Through Expectation of Function of Order Statistics, Mathematics and Statistics 2014 2(6): 205-213.
8	Bhatt Milind B. (2015), Characterization of Power Function Distribution through Expectation of Function of Order Statistics, Mathematics and Statistics, 2015, 3(6): 146-150.
9	Bhatt Milind B. (2015), Characterization of one-truncation parameter family of distributions through expectation, ProbStat Forum, Volume 08, January 2015, Pages 34{44}.

FINAL Report of the work done on the Major Research Project No. 42-39/2013(SR)

FINAL REPORT : MY CONTRIBUTION TO SOCIETY

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

10	Bhatt Milind B. (2016), Characterization of Pareto Distribution Through Expectation of Function of Order Statistics, Journal of Statistical Science and Application, August 2016, Vol. 4, No. 07-08, 196-203.
11	Bhatt Milind B. (2017), Characterization of Pareto distribution through expectation, Research Journal of Mathematical and Statistical Sciences, February (2017), Vol. 5(2), 1-4.
12	

(5) Award : International Best oral presentation

"Characterization of one truncation parameter family of distributions through expectation"

5th International Science Congress (ISC-2015) Conference at Tribhuvan University, Kathmandu, Nepal, organized by International Science Congress Association

(6) Field work :

Field work : 1 16-12-2013 to 19-12-2013, 4 days Field work of MRP for literature survey and discussion with Prof. David D. Hanagal, Department of Statistics University of Pune.

Field work : 2 26-12-2013 to 01-01-2014, 7 days Field work of MRP for literature survey and discussion with Dr. V. H. Bajaj, Professor and Head, Department of statistics Dr. Babasaheb Ambedkar Marathwada University, Aurangabad.

Field work : 3 1-11-2014 to 3-11-2014, 3 days Field work of MRP for literature survey and discussion with Prof j p singh joorel, Professor and Head, Department of statistics University of Jammu.

(6) Short term course

Short term course : 1 19-10-2013 to 24-10-2013, 1 Weeks UGC Sponsor Short-term Course on "Disaster Management", A.S.C., S. P. Univ., V. V. Nagar.

FINAL Report of the work done on the Major Research Project
No. 42-39/2013(SR)

FINAL REPORT : MY CONTRIBUTION TO SOCIETY

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

Short term course : 2 02-12-2013 to 07-12-2013, 1 Weeks UGC Sponsor Short-term Course on "Research Methodology", A.S.C., S. P. Univ., V. V. Nagar

ANNEXURE-III

P G DEPARTMENT OF STATISTICS
SARDAR PATEL UNIVERSITY
VALLABH VIDYA NAGAR-388120
GUJARAT

STATEMENT OF EXPENDITURE IN RESPECT OF MAJOR RESEARCH PROJECT

1. Name of Principal Investigator **Bhatt Milind Bhanuprasad**
2. Deptt. of Principal Investigator **Department of Statistics**

   University/College **Sardar Patel University**
3. UGC approval Letter No. and Date **F.42-39/2013(SR), dated 12-03-2013**
4. Title of the Research Project **Inferential problems on Non-regular (Truncation Parameter) Family of Distributions**
5. Effective date of starting the project **1-4-2013**
6. a. Period of Expenditure: From **1-4-2013 to 31-3-2017**
7. b. Details of Expenditure

S.No.	Item	Amount Approved (Rs.)	Expenditure Incurred (Rs.)
i.	Books & Journals	4,00,000.00	399614.00
ii.	Equipment	Nill	Nill
iii.	Contingency	1,50,000.00 - 30,000.00 Total 1,20,000.00	119700.00
iv.	Field Work/Travel (Give details in the	50,000.00 + 30,000.00 Total 80,000.00	71960.00

ANNEXURE-III

	proforma at Annexure-IV).		
v.	Hiring Services		
vi.	Chemicals & Glassware		
vii.	Overhead	67,800.00	67,800.00
viii.	Any other items (Please specify)		
TOTAL		**6,67,800.00**	**659074.00**

c . Staff **Mr. Hardikkumar Jashbhai Patel**

Date of Appointment **02-09-2013 TO February - 2014**

Staff **Mr. Kapil J Maheshwari**

Date of Appointment **11-10-2014 TO October – 2015**

S.No	Items	From	To	Amount Approved (Rs.)	Expenditure incurred (Rs.)
1.	Honorarium to PI (Retired Teachers) @ Rs. 18,000/-p.m.				
2.	**Project fellow:** i) **Mr. Hardikkumar Jashbhai Patel**	02-09-2013	14-12-2013		69533.00
	ii) **Mr. Kapil J Maheshwari**	11-10-2014	17-10-2015	**5,28,000.00**	178061.00

Final Report of the work done on the Major Research Project No. 42-39/2013(SR)

ANNEXURE-III

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

					247594.00

1. It is certified that the appointment(s) have been made in accordance with the terms and conditions laid down by the Commission.
2. If as a result of check or audit objection some irregularly is noticed at later date, action will be taken to refund, adjust or regularize the objected amounts.
3. Payment @ revised rates shall be made with arrears on the availability of additional funds.
8. It is certified that the grant of Rs. **8,31,800.00** (Rupees **Eight lac thirty one thousand eight hundrad** only) received from the University Grants Commission under the scheme of support for Major Research Project entitled **Inferential problems on Non-regular (Truncation parameter) family of istributions**
4. vide UGC letter No. F. **42-39/2013(SR)** dated **12-03-2013**

has been partly utilized for the purpose for which it was sanctioned and in accordance with the terms and conditions laid down by the University Grants Commission.

SIGNATURE OF PRINCIPAL INVESTIGATOR

REGISTRAR

Final Report of the work done on the Major Research Project No. 42-39/2013(SR)

Annexure - IV

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

P G DEPARTMENT OF STATISTICS

SARDAR PATEL UNIVERSITY

VALLABH VIDYA NAGAR-388120

GUJARAT

STATEMENT OF EXPENDITURE INCURRED ON FIELD WORK

Name of the Principal Investigator:

Name of the Place visited	Duration of the Visit		Mode of Journey	Expenditure Incurred (Rs.)
	From	To		
Department of Statistics, University of Kashmir, Srinagar,J&K, India	21-10-2013	23-10-2013	AIR	10446.00
Karunya University, Coimbatore, Tamil Nadu, India	08-10-2013	09-10-2013	AIR	13505.00
Department of Statistics, University of Pune	16-12-2014	16-12-2014	AIR	16070.00
Department of Statistics, Dr. B.A.M. University, Aurangabad	25-12-2014	31-12-2014	AIR	7241.00
Department of Statistics, University of Jammu	1-11-2014 to	3-11-2014	AIR	21265.00
Project fellow Interview			Road	970.00 1768.00 695.00

TOTAL 71960.00

Certified that the above expenditure is in accordance with the UGC norms for Major Research Projects.

SIGNATURE OF PRINCIPAL INVESTIGATOR

REGISTRAR/

DEPARTMENT OF STATISTICS
UGC-SAP DRS-I
UGC - INNOVATIVE PROGRAMME
DST-FIST SUPPORTED DEPARTMENT
Phone : 02692 226871 (Head)
02692 226881 (Office)
Fax : 02692 226871
E-mail : dstat_spu@yahoo.com
Website : www.spuvvn.edu/pgd/statistics/index.html
Sardar Patel University, Vallabh Vidyanagar – 388 120 Gujarat, India

UGC FILE NO.F. MRP 41-39/2013(SR) (HRP) YEAR OF COMMENCEMENT 01/04/2013

TITLE OF THE PROJECT :

1.	Name of the Principal Investigator	Dr. Milind Bhanuprashad Bhatt
2.	Name of the University	Post Graduate Department of statistics, Sardar Patel University, Vallabh Vidyanagar
3.	Name of the Research Personnel appointed	Hardikkumar Jashbhai Patel
4.	Academic qualification	(see below)
5.	Date of joining	02/09/2013
6.	Date of Birth of Research Personnel	01/07/1990
7.	Amount of HRA, if drawn	-
8.	Number of candidate applied for the post	02

Sr.No.	Qualifications	Year	Marks	%age
1.	M.Sc	2013	-	56.6
2.	M.Phil	-	-	-
3.	Ph.D	-	-	-

CERTIFICATE

This is to certify that all rules and regulations of UGC Major Research Project outlined in the guidelines have been followed. Any lapse on the part of the University will liable to terminate of said UGC project.

M. B. Bhatt
Principal Investigator

Head of Dept.
Head
Department of Statistics
Sardar Patel University
Vallabh Vidyanagar-388120.

Registrar/Principal
I/c. Registrar
Sardar Patel University
Vallabh Vidyanagar

DEPARTMENT OF STATISTICS
UGC-SAP DRS-I
UGC – INNOVATIVE PROGRAMME
DST-FIST SUPPORTED DEPARTMENT

Phone : 02692 226871 (Head)
02692 226881 (Office)
Fax : 02692 226871
E-mail : dstat_spu@yahoo.com
Website : www.spuvvn.edu/pgd/statistics/index.html

Sardar Patel University, Vallabh Vidyanagar – 388 120 Gujarat, India

Annexure-VI

UGC FILE NO.F. 42-39/2013(HRP) YEAR OF COMMENCEMENT 2 0 1 3 2 0 1 4

TITLE OF THE PROJECT: "Inferential problems on Non-regular (Truncation parameter) family of distributions"

1.	Name of the Principle Investigator:	Prof./Milind B. Bhatt (Assistant Professor)				
2.	Name of the University/college	P.G. Department of Statistics, Sardar Patel University, Vallabh Vidyanagar				
3.	Name of the Research Personnel Appointed	Mr. Kapil Javaharlal Maheshwari				
4.	Academic qualification	S.No.	Qualifications	Year	Marks	%age
		1.	M.A/M.Sc./M.Tech	2014	-	67.6
		2.	M.Phil			
		3.	Ph.D.			
5.	Date of joining	11/10/2014				
6.	Date of Birth of Research Personnel	20/03/1992				
7.	Amount of HRA, if drawn	-				
8.	Number of Candidate applied for the Post	2				

CERTIFICATE

This is to certify that all the rules and regulations of UGC Major Research Project outlined in the guidelines have been followed. Any lapse on the part of the University will liable to terminate of said UGC project.

M B Bhatt
Principle Investigator

Head of Department
Head
Department of Statistics
Sardar Patel University
Vallabh Vidyanagar

Registrar/Principal

SARDAR PATEL UNIVERSITY V.V. NAGAR

Final Report of the work done on the Major Research Project No. 42-39/2013(SR)

ANNEXURE-VIII

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

Annexure -VIII
UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.

Annual/Final Report of the work done on the Major Research Project.
(Report to be submitted within 6 weeks after completion of each year)

1 Project report No. 1st /2nd /3rd/**Final**

2 UGC Reference No.F. **42-39/2013(SR), dated 12-03-2013**

3 Period of report : **from 01-04-2015 to 31-03-2017**

4 Title of research project : **Inferential problems on Non-regular (Truncation parameter) family of distributions**

5 (a) Name of the Principal Investigator : **Bhatt Milind B**

(b) Deptt. **Statistics**

(c) University/College where work has progressed : **Sardar Patel University**

6 Effective date of starting of the project : **29-05-2013**

7 Grant approved and expenditure incurred during the period of the report:

(a) Total amount approved **Rs. 11.95,800.00**

* (b) Total expenditure **Rs. 906668.00**

* (c) Report of the work done: (Please attach a separate sheet) : Encl-

i Brief objective of the project :

Provide new path braking unified approach "identity of distribution and equality of expectation of function of random variable, function of order statistics" to characterize Negative Exponential Distribution, Power-Function Distribution, Pareto Distribution, Uniform Distribution and generalize Uniform Distribution through expectation and through expectation of function of order statistics, over and above other approaches available in statistics litterer such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy, independence of suitable function of order statistics, linear relation of conditional expectation, recurrence relations between expectations of function of order statistics, distributional properties of exponential distributional, record valves, lower record statistics, product of order statistics and Lorenz curve, etc.

7 (c) ii Work done so far and results achieved and publications, if any, resulting :

SR. NO.	Title of the published paper
1	Bhatt Milind B. (2013), Characterization of Negative Exponential Distribution through Expectation, Open Journal of Statistics, 2013, 3, 367-369,
2	Bhatt Milind B. (2013), Characterization of Power-Function Distribution through Expectation, Open Journal of Statistics, 2013, 3, 441-443.
3	Bhatt Milind B. (2014), Characterization of Negative Exponential Distribution Through Expectation of Function of Order Statistics, Journal of Statistical Science and Application 2 (2014) 79-84
4	Bhatt Milind B. (2014), Characterization of Uniform Distribution u (0,) through Expectation, Research Journal of Mathematical and Statistical Sciences, (2014) February Vol. 2(2), 16-19.
5	Bhatt Milind B. (2014), Characterization of Uniform Distribution through Expectation of Function of Order Statistics, Research Journal of Mathematical and Statistical Sciences, August (2014), Vol. 2(8), 10-14.
6	Bhatt Milind B. (2014), Characterization of Generalized Uniform Distribution through Expectation, Open Journal of Statistics, 2014, 4, 563-569.
7	Bhatt Milind B. (2014), Characterization of Generalized Uniform Distribution Through Expectation of Function of Order Statistics, Mathematics and Statistics 2014 2(6): 205-213.
8	Bhatt Milind B. (2015), Characterization of Power Function Distribution through Expectation of Function of Order Statistics, Mathematics and Statistics, 2015, 3(6): 146-150.

9 Bhatt Milind B. (2015), Characterization of one-truncation parameter family of distributions through expectation, ProbStat Forum, Volume 08, January 2015, Pages 34{44}.

10 Bhatt Milind B. (2016), Characterization of Pareto Distribution Through Expectation of Function of Order Statistics, Journal of Statistical Science and Application, August 2016, Vol. 4, No. 07-08, 196-203.

11 Bhatt Milind B. (2017), Characterization of Pareto distribution through expectation, Research Journal of Mathematical and Statistical Sciences, February (2017), Vol. 5(2), 1-4.

12

7 (c) iii Has the progress been according to original plan of work and towards achieving the objective. if not, state reasons : **Yes.**

7 (c)

iv Please indicate the difficulties, if any, experienced in implementing the project : **No.**

7 (c) v If project has not been completed, please indicate the approximate time by which it is likely to be completed. A summary of the work done for the period (Annual basis) may please be sent to the Commission on a separate sheet : **Note applicable. Project has been completed.**

7 (c) vi If the project has been completed, please enclose a summary of the findings of the study. One bound copy of the final report of work done may also be sent to University Grants Commission.

(1) Project has been completed.

(2) Bound copy of the final report of work done may also be sent to University Grants Commission.

7 (c) vii Any other information which would help in evaluation of work done on the project. At the completion of the project, the first report should indicate the output, such as (a) Manpower trained (b) Ph. D. awarded (c) Publication of results (d) other impact, if any **Yes.**

Twelve Publications.

Final Report of the work done on the Major Research Project
No. 42-39/2013(SR)

ANNEXURE-VIII

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

SIGNATURE OF THE PRINCIPAL INVESTIGATOR

SIGNATURE OF THE CO-INVESTIGAT : **No CO-INVESTIGAT**

REGISTRAR/PRINCIPAL

Final Report of the work done on the Major Research Project
No. 42-39/2013(SR)

ANNEXURE-IX

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002

PROFORMA FOR SUBMISSION OF INFORMATION AT THE TIME OF SENDING THE FINAL REPORT OF THE WORK DONE ON THE PROJECT

1 Title of the Project : **INFERENTIAL PROBLEMS ON NON-REGULAR (TRUNCATION PARAMETER) FAMILY OF DISTRIBUTIONS**

2 NAME AND ADDRESS OF THE PRINCIPAL INVESTIGATOR :

(1) NAME OF PRINCIPAL INVESTIGATOR: **Bhatt Milind B**

(2) ADDRESS : **Department of Statistics, Sardar Patel University, Vallavbh Vidyanagar-388120,**
Dist : Anand, GUJARAT.

3 NAME AND ADDRESS OF THE INSTITUTION :

(1) NAME OF THE INSTITUTION : **Department of Statistics**

(2) ADDRESS : **Department of Statistics, Sardar Patel University, Vallavbh Vidyanagar-388120, Dist : Anand, GUJARAT**

4 UGC APPROVAL LETTER NO. AND DATE :

(1) UGC APPROVAL LETTER NO: F. **42-39/2013(SR)**

(2) DATE : **dated 12-03-2013**

5 DATE OF IMPLEMENTATION : **29-05-2013**

6 TENURE OF THE PROJECT : **THREE YEARS**

7 TOTAL GRANT ALLOCATED : **Rs. 11.95,800.00**

Final Report of the work done on the Major Research Project No. 42-39/2013(SR)

ANNEXURE-IX

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

8 TOTAL GRANT RECEIVED : **Rs. 831800.00**

9 FINAL EXPENDITURE : **Rs. 906668.00**

10 Title of the Project : **Inferential problems on Non-regular (Truncation parameter) family of distributions**

11 OBJECTIVES OF THE PROJECT :

Provide new path braking unified approach "identity of distribution and equality of expectation of function of random variable, function of order statistics" to characterize Negative Exponential Distribution, Power-Function Distribution, Pareto Distribution, Uniform Distribution and generalize Uniform Distribution through expectation and through expectation of function of order statistics.

12 WHETHER OBJECTIVES WERE ACHIEVED (GIVE DETAILS) : **Yes.**

Over and above other approaches available in statistics litterer for characterization of distributions such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy, independence of suitable function of order statistics, linear relation of conditional expectation, recurrence relations between expectations of function of order statistics, distributional properties of exponential distributional, record valves, lower record statistics, product of order statistics and Lorenz curve, etc.

New path braking unified approach "identity of distribution and equality of expectation of function of random variable, function of order statistics" to characterize Negative Exponential Distribution, Power-Function Distribution, Pareto Distribution, Uniform Distribution and generalize Uniform Distribution through expectation and through expectation of function of order statistics.

13 ACHIEVEMENTS FROM THE PROJECT

1 Bhatt Milind B. (2013), Characterization of Negative Exponential Distribution through Expectation, Open Journal of Statistics, 2013, 3, 367-369,

2 Bhatt Milind B. (2013), Characterization of Power-Function Distribution through Expectation, Open Journal of Statistics, 2013, 3, 441-443.

3 Bhatt Milind B. (2014), Characterization of Negative Exponential Distribution Through Expectation of Function of Order Statistics, Journal of Statistical Science and Application 2 (2014) 79-84

4 Bhatt Milind B. (2014), Characterization of Uniform Distribution $U(0,\theta)$through Expectation, Research Journal of Mathematical and Statistical Sciences, (2014) February Vol. 2(2), 16-19.

5 Bhatt Milind B. (2014), Characterization of Uniform Distribution through Expectation of Function of Order Statistics, Research Journal of Mathematical and Statistical Sciences, August (2014), Vol. 2(8), 10-14.

6 Bhatt Milind B. (2014), Characterization of Generalized Uniform Distribution through Expectation, Open Journal of Statistics, 2014, 4, 563-569.

7 Bhatt Milind B. (2014), Characterization of Generalized Uniform Distribution Through Expectation of Function of Order Statistics, Mathematics and Statistics 2014 2(6): 205-213.

8 Bhatt Milind B. (2015), Characterization of Power Function Distribution through Expectation of Function of Order Statistics, Mathematics and Statistics, 2015, 3(6): 146-150.

9 Bhatt Milind B. (2015), Characterization of one-truncation parameter family of distributions through expectation, ProbStat Forum, Volume 08, January 2015, Pages 34{44}.

10 Bhatt Milind B. (2016), Characterization of Pareto Distribution Through Expectation of Function of Order Statistics, Journal of Statistical Science and Application, August 2016, Vol. 4, No. 07-08, 196-203.

11 Bhatt Milind B. (2017), Characterization of Pareto distribution through expectation, Research Journal of Mathematical and Statistical Sciences, February (2017), Vol. 5(2), 1-4.

12

14 SUMMARY OF THE FINDINGS (IN 500 WORDS) :

Unified approach "identity of distribution and equality of expectation of function of random variable, function of order statistics" to characterize Negative Exponential Distribution, Power-Function Distribution, Pareto Distribution, Uniform Distribution and generalize Uniform Distribution through expectation and through expectation of function of order statistics, Over and above other approaches available in statistics litterer for characterization of distributions such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy, independence of suitable function of order statistics, linear relation of conditional expectation, recurrence relations between expectations of function of order statistics, distributional properties of exponential distributional, record valves, lower record statistics, product of order statistics and Lorenz curve, etc.

15 CONTRIBUTION TO THE SOCIETY (GIVE DETAILS) :

Path braking new unified approach "identity of distribution and equality of expectation of function of random variable, function of order statistics" to characterize Negative Exponential Distribution, Power-Function Distribution, Pareto Distribution, Uniform Distribution and generalize Uniform Distribution through expectation and through expectation of function of order statistics.

16 WHETHER ANY PH.D. ENROLLED/PRODUCED OUT OF THE PROJECT : **No.**

17 NO. OF PUBLICATIONS OUT OF THE PROJECT (PLEASE ATTACH) : **Yes,**
All Twelve Publications attach.

PRINCIPAL INVESTIGATOR

REGISTRAR

Final Report of the work done on the Major Research Project No. 42-39/2013(SR)

ANNEXURE-IX---OF---(17)

NO. OF PUBLICATIONS OUT OF THE PROJECT (PLEASE ATTACH) : Yes

Bhatt Milind Bhanuprasad, Department of Statistics Sardar Patel University, Vallabh Vidya Nagar
Project title : Inferential Problems on Non-regular (Truncation Parameter) Family of Distributions

Sr. No.	Title of the published paper	Page No.
1	Characterization of Negative Exponential Distribution through Expectation.	367-379
2	Characterization of Negative Exponential Distribution through Expectation of Function of Order Statistics,	79-84
3	Characterization of Pareto distribution through expectation.	1-4
4	Characterization of Pareto distribution through expectation of Function of Order Statistics.	196-203
5	Characterization of Power-Function Distribution through Expectation.	441-443
6	Characterization of Power-Function Distribution through Expectation of Function of Order Statistics.	146-150
7	Characterization of Uniform Distribution $U(0,\theta)$through Expectation.	16-19
8	Characterization of Uniform Distribution through Expectation of Function of Order Statistics.	10-14
9	Characterization of Generalized Uniform Distribution through Expectation.	563-569
10	Characterization of Generalized Uniform Distribution Through Expectation of Function of Order Statistics.	205-213
11	Characterization of one-truncation parameter family of distributions through expectation.	34-44
12	Characterization of one-truncation parameter family of distributions through expectation of Function of Order Statistics.	51-62

Open Journal of Statistics, **2013, 3, 367-369**
http://dx.doi.org/10.4236/ojs.2013.35042 Published Online October 2013 (http://www.scirp.org/journal/ojs)

Characterization of Negative Exponential Distribution through Expectation*

Milind Bhatt B.
Department of Statistics, Sardar Patel University, Vallabh Vidyanagar, India
Email: bhattmilind_b@yahoo.com

Received May 4, 2013; revised June 4, 2013; accepted June 11, 2013

ABSTRACT

For characterization of negative exponential distribution one needs any arbitrary non-constant function only in place of approaches such as identical distributions, absolute continuity, constant regression of order statistics, continuity and linear regression of order statistics, non-degeneracy etc. available in the literature. Path breaking different approach for characterization of negative exponential distribution through expectation of non-constant function of random variable is obtained. An example is given for illustrative purpose.

Keywords: Characterization; Negative Exponential Distribution

1. Introduction

Knowing characterizing property may provide unexpectedly accurate information about distributions and one can recognize a class of distributions before any statistical inference is made. This feature of characterization of probability distributions is peculiar to characterizing property and attracted attention of both theoretician and applied workers but there is no general theory of it.

Various approaches were used for characterization of negative exponential distribution. Among many other people, Fisz [1], Tanis [2], Rogers [3] and Fergusion [4] used properties of identical distributions, absolute continuity, constant regression of adjacent order statistics, linear regression of adjacent order statistics of random variables and characterized negative exponential distribution. Using independent and non-degenerate random variables Fergusion ([5,6]) and Crawford [7] characterized negative exponential distribution. Linear regression of two adjacent record values used by Nagaraja ([8,9],) were different from two conditional expectations, conditioned on a non-adjacent order statistics used by Khan [10] to characterize negative exponential distribution.

In this research note section 2 is devoted for characterization based on identity of distribution and equality of expectation function randomly variable for a negative exponential distribution with probability density function (pdf).

$$f(x;\theta)=\begin{cases} e^{-(x-\theta)}; & a<\theta<x<b \\ 0; & \text{otherwise} \end{cases} \tag{1.1}$$

where $-\infty \le a < b \le \infty$ are known as constants, e^{-x} is positive absolute continuous function and e^{θ} is everywhere differentiable function. Since derivative of e^{θ} is positive, the range is truncated by θ from left $e^{-b}=0$.

2. Characterization

Theorem 2.1 Let X be a random variable with distribution function F. Assume that F is continuous on the interval (a,b), where $-\infty \le a < b \le \infty$. Let $\phi(X)$ and $g(X)$ be two distinct differentiable and intregrable functions of X on the interval (a,b) where $-\infty \le a < b \le \infty$ and moreover $g(X)$ be non constant. Then

$$E\left[g(X)-(\mathrm{d}/\mathrm{d}X)g(X)\right]=g(\theta) \tag{2.1}$$

is the necessary and sufficient condition for pdf $f(x;\theta)$ of F to be $f(x;\theta)$ defined in (1.1).

Proof Given $f(x;\theta)$ defined in (1.1), for necessity of (2.1) if $\phi(X)$ is such that $g(\theta)=E\left[\phi(X)\right]$ where $g(\theta)$ is differentiable function then

$$g(\theta)=\int_{\theta}^{b}\phi(X)f(x;\theta)\mathrm{d}x \tag{2.2}$$

Differentiating with respect to θ on both sides of (2.2), replacing X for θ and simplifying one gets

$$\phi(X)=\left[g(X)-(\mathrm{d}g(X)/\mathrm{d}x)\right] \tag{2.3}$$

*This work is supported by UGC Major Research Project No: F.No.42-39/2013(SR), dated 12-3-2013.

which establishes necessity of (2.1). Conversely given (2.1), let $k(x;\theta)$ be such that

$$g(\theta)=\int_\theta^b\left[g(x)-\mathrm{d}g(x)/\mathrm{d}x\right]k(x;\theta)\,\mathrm{d}x \qquad (2.4)$$

which can be rewritten as

$$g(\theta)=-\mathrm{e}^{\theta}\int_\theta^b\Big[\Big(g(x)+\left\{\mathrm{e}^{-x}/\left(\mathrm{de}^{-x}/\mathrm{d}x\right)\right\}\left(\mathrm{d}g(x)/\mathrm{d}x\right)\Big) k(x;\theta)/\left(-\mathrm{e}^{\theta}\right)\Big]\mathrm{d}x \qquad (2.5)$$

which reduces to

$$g(\theta)=-\mathrm{e}^{\theta}\int_\theta^b\left[\mathrm{d}\left(\mathrm{e}^{-x}g(x)\right)/\mathrm{d}x\right] \left[k(x;\theta)/\left(-\mathrm{e}^{\theta}\right)\left\{\mathrm{d}\left(\mathrm{e}^{-x}\right)/\mathrm{d}x\right\}\right]\mathrm{d}x \qquad (2.6)$$

Hence

$$k(x;\theta)=\mathrm{e}^{\theta}\left(-\mathrm{de}^{-x}/\mathrm{d}x\right). \qquad (2.7)$$

Since e^{x} is increasing integrable and differentiable function on the interval (a,b) with $\mathrm{e}^{-b}=0$ the following identity holds

$$g(\theta)=-\mathrm{e}^{\theta}\int_\theta^b\left[\mathrm{d}\left\{\mathrm{e}^{-x}g(x)\right\}/\mathrm{d}x\right]\mathrm{d}x. \qquad (2.8)$$

Differentiating $\left\{\mathrm{e}^{-x}g(x)\right\}$ with respect to x and simplifying (2.8) after taking $-\mathrm{de}^{-x}/\mathrm{d}x$ as one factor, (2.8) reduces to

$$g(\theta)=\int_\theta^b\phi(x)k(x;\theta)\,\mathrm{d}x, \qquad (2.9)$$

where $\phi(X)$ is a function of X only derived in (2.3) and $k(x;\theta)$ is a function of x and θ only derived in (2.7).

Since e^{x} be increasing integrable and differentiable function on the interval (a,b) where $-\infty\le a<b\le\infty$ and since $-\mathrm{de}^{-x}/\mathrm{d}x$ is positive intregrable function on the interval (a,b) where $-\infty\le a<b\le\infty$ with $\mathrm{e}^{-b}=0$ and integrating (2.7) over the interval (θ,b) on both sides, one gets (2.7) as

$$k(x;\theta)=\mathrm{e}^{-\theta}\left(-\mathrm{de}^{-x}/\mathrm{d}x\right);a<\theta<x<b \qquad (2.10)$$

and

$$1=\int_\theta^b k(x;\theta)\,\mathrm{d}x.$$

Substituting $\mathrm{de}^{-x}/\mathrm{d}x$ in $k(x;\theta)$ derived in (2.10), $k(x;\theta)$ reduces to $f(x;\theta)$ defined in (1.1) which establishes sufficiency of (2.1).

Remark 2.1 Using $\phi(X)$ derived in (2.3), the $f(x;\theta)$ given in (1.1) can be determined by

$$M(X)=\left(\mathrm{d}g(x)/\mathrm{d}x\right)/\left(\phi(x)-g(X)\right) \qquad (2.11)$$

and pdf is given by

$$f(x;\theta)=-\left(\mathrm{d}T(x)/\mathrm{d}x\right)/T(\theta) \qquad (2.12)$$

where $T(x)$ is decreasing function for $-\infty\le a<b\le\infty$ with $T(b)=0$ such that it satisfies

$$M(X)=\mathrm{d}\left[\log\left\{T(X)\right\}\right]/\mathrm{d}X. \qquad (2.13)$$

Illustrative Example: Using method described in the remark characterization of negative exponential distribution through survival function $g(\theta)=\bar{F}(\theta)=\mathrm{e}^{-(t-\theta)}$ is illustrated.

$$g(X)=\mathrm{e}^{-(t-X)}$$

$$\phi(X)=g(X)-\mathrm{d}g(X)/\mathrm{d}X=\mathrm{e}^{-(t-X)}-\mathrm{e}^{-(t-X)}=0$$

$$M(X)=\left(\mathrm{d}g(X)/\mathrm{d}X\right)/\left(f(X)-g(X)\right)=-1$$

$$\mathrm{d}\left\{\log\left(\mathrm{e}^{-X}\right)\right\}/\mathrm{d}X=M(X)=-1$$

$$\therefore T(X)=\mathrm{e}^{-X}$$

$$f(x;\theta)=\mathrm{e}^{-(x-\theta)};a<\theta<x<b$$

3. Conclusion

To characterize pdf defined in (1.1) one needs any arbitrary non-constant function of X which should only be differentiable and integrable.

REFERENCES

[1] M. Fisz, "Characterization of Some Probability Distributions," *Skandinavisk Aktuarietidskrift*, Vol. 41, No. 1-2, 1958, pp. 65-70.

[2] E. Tanis, "Linear Forms in the Order Statistics from an Exponential Distribution," *The Annals of Mathematical Statistics*, Vol. 35, No. 1, 1964, pp. 270-276.

[3] G. S. Rogers, "An Alternative Proof of the Characterization of the Density AxB," *The American Mathematical Monthly*, Vol. 70, No. 8, 1963, pp. 857-858.

[4] T. S. Ferguson, "A Characterization of the Negative Exponential Distribution," *The Annals of Mathematical Statistics*, Vol. 35, 1964, pp. 1199-1207.

[5] T. S. Ferguson, "A Characterization of the Geometric Distribution," *The American Mathematical Monthly*, Vol. 72, No. 3, 1965, pp. 256-260.

[6] T. S. Ferguson, "On Characterizing Distributions by Properties of Order Statistics," *Sankhyā: The Indian Journal of Statistics, Series A* (1961-2002), Vol. 29, No. 3, 1967, pp. 265-278.

[7] G. B. Crawford, "Characterizations of Geometric and Exponential Distributions," *The Annals of Mathematical Statistics*, Vol. 37, No. 6, 1966, pp. 1790-1795.

[8] H. N. Nagaraja, "On a Characterization Based on Record Values," *Australian Journal of Statistics*, Vol. 19, 1977, pp. 70-73.

[9] H. N. Nagaraja, "Some Characterization of Continuous Distributions Based on Adjacent Order Statistics and Re-

cord Values," *Sankhy, Series A*, Vol. 50, No. 1, 1988, pp. 70-73.

[10] A. H. Khan, M. Faizan and Z. Haque, "Characterization of Probability Distributions through Order Statistics," *Prob Stat Forum*, Vol. 2, 2009, pp. 132-136.

Journal of Statistical Science and Application 2 (2014) 79-84

DAVID PUBLISHING

Characterization of Negative Exponential Distribution Through Expectation of Function of Order Statistics

Bhatt Milind. B

Department of Statistics, Sardar Patel University, Vallabh Vidyanagar , Dist. Anand, Gujarat, India

For characterization of negative exponential distribution one needs any arbitrary non constant function only in place of approaches such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy etc. available in the literature. Recently Bhatt characterized negative exponential distribution through expectation of non constant function of random variable. Attempt is made to extend the characterization of negative exponential distribution through expectation of any arbitrary non constant function of order statistics.

Introduction

Let $X_1, X_2, \dots, X_n$ be independent identically distributed random variables with absolutely continuous distribution F and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the set of corresponding order statistics. For $n = 2$ Fisz (1958) assumed that random variables X_1 and X_2, were positive and used ratio instead of difference and proved that X_1 and X_2 have negative exponential distribution if and only if $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent. Tanis (1964) made Fisz's result arbitrary for any finite n, that is, $X_{(1)}$ and $\sum_{i=2}^{n}(X_{(i)} - X_{(1)})$ are independent under the same setup (identical distributions and absolute continuity) and the same method (derive results for exponent of negative of assumed positive variables using ratio instead of difference).

Under the same setup assuming independence of positive difference of adjacent order statistics; $(X_{(m+1)} - X_{(m)})$ and $X_{(m)}$; $m = 2, 3, \dots, n$, Rogers (1963) showed that if the regression of $(X_{(m+1)} - X_{(m)})$ on $X_{(m)}$ is constant then distribution F is negative exponential. replacing Fisz's assumptions of absolute continuity and Rogers's assumptions of constancy of regression of $(X_{(m+1)} - X_{(m)})$ on $X_{(m)}$ by continuity and linear regression of $X_{(m+1)}$ on $X_{(m)}$, Fergusion (1967) generalizd the Roger's result and noted that it is not known which distribution would be characterized if non-adjacent order statistics were considered. Closely related to Fisz'sresults, Fergusion (1964, 1965) and Crawford (1966) considered stronger assumptions of independence and non-degeneracy for X_1, X_2 in place of identical distribution and absolute continuity, derived characterization for negative exponential distribution and geometric distribution. Srivastave (1967) added one more characterization to existing set of characterization of negative

Nagaraja (1977, 1988) obtained characterization results based on linear regression of two adjacent record valuesand noted that Fergusion's remark for non-adjacent order statistics holds for record values too. Khan (2009) characterized family of continuous probability distributions through the difference of two conditional expectations, conditioned on anon-adjacent order statistics which include negative exponential distribution.

Corresponding author: Bhatt Milind. B, Ph.D, research field: Statistical inference, Department of Statistics, Sardar Patel University, India. Email: bhattmilind_b@yahoo.com.

Using identity of distribution and equality of expectation of function of a random variable, Bhatt (2013) derived characterization of negative exponential distribution with probability density function (pdf)

$$f_X(x,\theta) = \begin{cases} e^{-(x-\theta)}\,;\ \theta < x < \infty; \\ 0\,;\text{otherwise,} \end{cases} \tag{1.1}$$

where $-\infty \leq a < b \leq \infty$ are known constants, e^{-x} is positive absolutely continuous function and e^{θ} is everywhere differentiable function. Since derivative of e^{θ} being positive and since range is truncated by θ from left $e^{-b} = 0$.

Aim of this research note is to extend Bhatt's result for expectation of order statistics. In section 2 negative exponential distribution with pdf (1.1) is characterized by using function of first order statistic. Section 3 is devoted for illustrativeexamples.

Characterization

Theorem 2.1: Let $X_1, X_2, \dots, X_n$ be a random sample of size n from distribution function F. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the corresponding set of order statistics. Assume that F is continuous on the interval (a,b), where $-\infty \leq a < b \leq \infty$. Let $\phi(X_{1:n})$ and $g(X_{1:n})$ be two distinct differentiable and intregrable functions of first order statistic; $X_{1:n}$, on the interval (a,b), where $-\infty \leq a < b \leq \infty$ and moreover $g(X_{1:n})$ be non-constant of $X_{1:n}$. Then

$$E\left[g(X_{1:n}) - \left(\frac{1}{n}\right)\frac{d}{dX_{1:n}} g(X_{1:n})\right] = g(\theta) \tag{2.1}$$

is the necessary and sufficient condition for pdf $f(x,\theta)$ of F to be $f(x,\theta)$ defined in (1.1).

Proof. Given $f(x,\theta)$ defined in (1.1), for necessity of (2.1) if $\phi(X_{1:n})$ is such that $g(\theta) = E[\phi(X_{1:n})]$ where $g(\theta)$ is differentiable function then using $f(x_{1:n},\theta)$; pdf of first order statistics one gets

$$g(\theta) = \int_\theta^b \phi(x_{1:n}) f(x_{1:n},\theta)\, dx_{1:n} \tag{2.2}$$

Differentiating with respect to θ on both sides of (2.2), replacing $X_{1:n}$ for θ and simplifying the result one gets

$$\phi(X_{1:n}) = g(X_{1:n}) - \left(\frac{1}{n}\right)\frac{d}{dX_{1:n}} g(X_{1:n}) \tag{2.3}$$

which establishes necessity of (2.1). Conversely given (2.1), if $k(x_{1:n},\theta)$ is pdf of first order statistics; $X_{1:n}$ of random variable X then and one gets $g(\theta)$ as

$$g(\theta) = \int_\theta^b \left[g(x_{1:n}) - \left(\frac{1}{n}\right)\frac{d}{dx_{1:n}} g(x_{1:n})\right] k(x_{1:n},\theta)\, dx_{1:n} \tag{2.4}$$

Since $e^{nx_{1:n}}$ is increasing integrable and differentiable function on the interval (a,b) with $e^{-b} = 0$ the following identity holds

$$g(\theta) \equiv -e^{n\theta} \int_{\theta}^{b} \left[\frac{d}{dx_{1:n}}\{e^{-nx_{1:n}} g(x_{1:n})\}\right] dx_{1:n} \tag{2.5}$$

Differentiating integrand of (2.5), $\{e^{-nx_{1:n}} g(x_{1:n})\}$ with respect to $x_{1:n}$ one gets

$$g(\theta) \equiv -e^{n\theta} \int_{\theta}^{b} \left[g(x_{1:n}) \frac{d}{dx_{1:n}} e^{-nx_{1:n}} + e^{-nx_{1:n}} \frac{d}{dx_{1:n}} g(x_{1:n})\right] dx_{1:n} \tag{2.6}$$

Taking $\left\{-\frac{d}{dx_{1:n}}(e^{-nx_{1:n}})\right\}$ as one factor, (2.6) reduces to

$$g(\theta) \equiv -e^{n\theta} \int_{\theta}^{b} \left[g(x_{1:n}) + \frac{e^{-nx_{1:n}}}{\frac{d}{dx_{1:n}} e^{-nx_{1:n}}} \frac{d}{dx_{1:n}} g(x_{1:n})\right] \frac{d}{dx_{1:n}} e^{-nx_{1:n}}\, dx_{1:n} \tag{2.7}$$

and after simplification one gets $g(\theta)$ as

$$g(\theta) \equiv \int_{\theta}^{b} \left[g(x_{1:n}) + \frac{e^{-nx_{1:n}}}{\frac{d}{dx_{1:n}} e^{-nx_{1:n}}} \frac{d}{dx_{1:n}} g(x_{1:n})\right] \left\{-e^{n\theta} \frac{d}{dx_{1:n}} e^{-nx_{1:n}}\right\} dx_{1:n} \tag{2.8}$$

Substituting $\frac{d}{dx_{1:n}} e^{-nx_{1:n}}$ in (2.8) one gets

$$g(\theta) \equiv \int_{\theta}^{b} \left[g(x_{1:n}) - \left(\frac{1}{n}\right) \frac{d}{dx_{1:n}} g(x_{1:n})\right] \left\{\frac{1}{n} e^{-n(x_{1:n}-\theta)}\right\} dx_{1:n} \tag{2.9}$$

From (2.4) and (2.9) by uniqueness theorem

$$k(x_{1:n}, \theta) = \frac{1}{n} e^{-n(x_{1:n}-\theta)}; \quad a < \theta < x_{1:n} < b \tag{2.10}$$

Since $e^{nx_{1:n}}$ be increasing integrable and differentiable function on the interval (a, b) where $-\infty \le a < b \le \infty$ with $e^{-b} = 0$ and integrating (2.10) over the interval (θ, b) on both sides, one gets

$1 = \int_{\theta}^{b} k(x_{1:n};\, \theta)\, dx_{1:n}$.

Hence from (2.10) $[k(x_{1:n};\theta)]_{n=1}$ reduces to $f(x;\theta)$ defined in (1.1) which establishes sufficiency of (2.1).

Remark 2.1

Using $\phi(X_{1:n})$ derived in (2.3), the $f(x,\theta)$ given in (1.1) can be determined by

$$M(X_{1:n}) = \frac{\frac{d}{dX_{1:n}} g(X_{1:n})}{\phi(X_{1:n}) - g(X_{1:n})} \tag{2.11}$$

and pdf is given by

$$f(x,\theta) = -\left[\frac{\frac{d}{dX_{1:n}}(T(X_{1:n}))}{T(\theta)}\right]_{n=1} \tag{2.12}$$

where $T(X_{1:n})$ is decreasing function for $-\infty \le a < b \le \infty$ with $T(b) = 0$ such that it satisfies

$$M(X_{1:n}) = \frac{d}{dX_{1:n}}\left[log\left(T(X_{1:n})\right)\right] \tag{2.13}$$

Remark 2.2 The theorem 2.1 for function of first order statistics also holds for random variable X when $(n = 1)$.

Illustrative Examples

Example. 3.1 Using method described in the remark characterization of negative exponential distribution through expectation of non constant function of order statistics $g(X_{1:n})$ as the uniformly minimum variance unbiased (UMVU) estimator $\hat{g}(\theta)$ and maximum likelihood estimator (MLE) $\tilde{g}(\theta)$ of $g(\theta)$ such as $\mu_1'(\theta)$, mean; $\mu_r'(\theta)$, r^{th} moment; e^{θ}, $e^{-\theta}$; $Q_p(\theta)$, p^{th} quantile; $F(t)$, distribution function; $\bar{F}(t)$, reliability function; $\lambda(t)$, hazard rat one gets $[\phi(X_{1:n}) - g(X_{1:n})]$ as given below.

$g(\theta)$	$g(X_{1:n})$	$\phi(X_{1:n}) - g(X_{1:n})$
$\mu_1'(\theta)$	$\widehat{\mu_1(\theta)} = X_{1:n} + 1 - \frac{1}{n}$	$-\frac{1}{n}$
	$\widetilde{\mu_1(\theta)} = X_{1:n} + 1$	$-\frac{1}{n}$
$\mu_r'(\theta)$	$\widehat{\mu_r(\theta)} = \left[\begin{array}{c}\sum_{i=0}^{r}\frac{r!}{(r-i)!}x_{1:n}^{r-i} \\ -\frac{1}{n}\sum_{i=0}^{r}\frac{r!}{(r-i-1)!}x_{1:n}^{r-i-1}\end{array}\right]$	$-\frac{1}{n}\left[\begin{array}{c}\frac{1}{n}\sum_{i=0}^{r}\frac{r!\,(r-i-1)}{(r-i-2)!}x_{1:n}^{r-i-2} \\ -\sum_{i=0}^{r}\frac{r!}{(r-i-1)!}x_{1:n}^{r-i-1}\end{array}\right]$
	$\widetilde{\mu_r(\theta)} = \sum_{i=0}^{r}\frac{r!}{(r-i)!}x_{1:n}^{r-i}$	$-\frac{1}{n}\sum_{i=0}^{r}\frac{r!}{(r-i-1)!}x_{1:n}^{r-i-1}$
e^{θ}	$\widehat{e^{\theta}} = e^{X_{1:n}}\left(1 - \frac{1}{n}\right)$	$-e^{-X_{1:n}}\left(1 - \frac{1}{n}\right)\frac{1}{n}$
	$\widetilde{e^{\theta}} = e^{X_{1:n}}$	$-\frac{e^{X_{1:n}}}{n}$
$e^{-\theta}$	$\widehat{e^{-\theta}} = e^{-X_{1:n}}\left(1 + \frac{1}{n}\right)$	$e^{-X_{1:n}}\left(1 + \frac{1}{n}\right)\frac{1}{n}$
	$\widetilde{e^{-\theta}} = e^{-X_{1:n}}$	$\frac{e^{-X_{1:n}}}{n}$
$Q_p(\theta)$	$\widehat{Q_p(\theta)} = -log(1-P) + X_{1:n} - \frac{1}{n}$	$-\frac{1}{n}$
	$\widetilde{Q_p(\theta)} = -log(1-P) + X_{1:n}$	$-\frac{1}{n}$
$F(t)$	$\hat{F}(t) = \left\{1 - e^{-(t-X_{1:n})}\left[1 - \frac{1}{n}\right]\right\}$	$\frac{1}{n}\left[1 - \frac{1}{n}\right]e^{-(t-X_{1:n})}$
	$\tilde{F}(t) = 1 - e^{-(t-X_{1:n})}$	$-\frac{1}{n}e^{-(t-X_{1:n})}$
$\bar{F}(t)$	$\hat{\bar{F}}(t) = e^{-(t-X_{1:n})}\left[1 - \frac{1}{n}\right]$	$-\frac{1}{n}\left[1 - \frac{1}{n}\right]e^{-(t-X_{1:n})}$
	$\tilde{\bar{F}}(t) = e^{-(t-X_{1:n})}$	$-\frac{1}{n}e^{-(t-X_{1:n})}$

Defining $M(X_{1:n})$ given in (2.11) and using $T(X_{1:n})$ as appeared in (2.13), the pdf (1.1) is characterized by (2.12).

Example. 3.2 In context of remark 2.2 the pdf $f(x,\theta)$ defined in (1.1) can be characterized through non constant functions of random variable X when $(n = 1)$

$$g_i(\theta) = \begin{cases} \theta + 1; for i = 1, Mean, \\ \sum_{k=0}^{r} \frac{r!}{(r-k)!} \theta^{r-k}; for i = 2, \text{r}^{\text{th}} \text{ raw moment,} \\ e^{\theta}; for i = 3, \\ e^{-\theta}; for i = 4, \\ -\log(1-p) + \theta; for i = 5, \text{p}^{\text{th}} \text{ quantile,} \\ 1 - e^{-(t-\theta)}; for i = 6, \text{distribution function at t,} \\ \text{e}^{-(t-\theta)}; for i = 7, \text{reliability function at t,} \\ 1; for i = 8, \text{hazard Function,} \end{cases}$$

by using

$$[\phi_i(X) - g_i(X)] = \begin{cases} -1; for i = 1, \text{Mean,} \\ -\sum_{k=0}^{r-1} \frac{r!}{(r-k-1)!} X^{r-k-1}; for i = 2, r^{th} \text{ raw moment,} \\ -e^{x}; for i = 3, \\ e^{-x}; for i = 4, \\ -1; for i = 5, p^{th} \text{ quantile,} \\ e^{-(t-x)}; for i = 6, \text{distribution function at } t, \\ -e^{-(t-x)}; for i = 7, \text{reliability function at } t, \end{cases}$$

and defining $M(X)$ given in (2.11) and using $T(X)$ as appeared in (2.13), for (2.12).

Note that as Hazard function for negative exponential distribution being constant it can not characterize pdf $f(x,\theta)$ given in (1.1).

Refference

Bhat, t M. B. (2013) Characterization of negative exponential distribution through expectation of function of random variable. To be appear in *Open Journal of Statistics*.

Crawford, G. B. (1966) : Characterizations of geometric and exponential distributions. *Ann. Math. Stat.*, 37, 1790-1795.

Fisz, M.(1958) : Characterization of some probability distributions. *Skand. Aktuarietidskr*, 41, 65-70.

Ferguson, T. S. (1964) : A characterization of the negative exponential distribution. *Ann. Math. Stat.*, 35, 1199-1207.

Ferguson, T. S. (1965) : A characterization of the geometric distribution. *Amer. Math. Monthly*, 72, 256-260.

Ferguson, T. S. (1967) :on characterizing distributions by properties Of order statistics : sankhyā: *the indian journal of statistics*, series a (1961-2002), vol. 29, no. 3 (sep.,1967), pp. 265-278.

Nagaraja, H. N.(1977) : On a characterization based on record values. *Australian Journal of Statistics*. 19 :70 - 73.

Nagaraja, H. N.(1988) : Some characterization of continuous distributions based on adjacent order statistics and record values, Sankhy_a, Ser.A, 50,70-73.

Rogers, G. S. (1963) : An alternative proof of the cnaracterization of the density AxB. Amer. Math. Monthly, 70, 857-858.

Srivastava, M. S.(1967) : A Characterization of the Exponential Distribution. *The American Mathematical Monthly*, Vol. 74, No. 4 (Apr.,1967), pp. 414-416.

Tanis, E. (1964) : Linear forms in the order statistics from an exponential distributer. *Ann. Math. Stat*, 35,

Research Journal of Mathematical and Statistical Sciences ISSN 2320-6047
Vol. **5(2)**, 1-4, February **(2017)** *Res. J. Mathematical and Statistical Sci.*

Characterization of pareto distribution through expectation

Bhatt Milind B.
Department of Statistics, Sardar Patel University, V.V. Nagar-388120, Gujarat, India
bhattmilind_b@yahoo.com

Available online at: www.isca.in, www.isca.me
Received 12th November 2016, revised 5th February 2017, accepted 10th February 2017

Abstract

In statistical inference we often encounter the situation, for instant 60 observations of random phenomena observed and one group of students fit normal distribution whereas other group fits log-normal distribution with almost same p-value. This is one of cases where characterization results provide navigation tools for correct direction for further study (research). One of the research materials which receives room in both pure (mathematics) and applied (probability) is characterization rules because they are boundary points of both. That is why mathematical community as well as probabilists and statisticians contribute research are of characterizations. Characterization is not concern to academician or researcher only who deals with foundation and application of probabilistic model building but also concern to operation research, behavior science, natural science, decision making process in industrial and engineering problem. For characterization of Pareto distribution one needs any arbitrary non constant function only by approach of identity of distribution and equality of expectation of function of random variable in place of approaches such as relation (linear) in (economic variation) reported and true income, independency of suitable function of order statistics, mean and the extreme observation of the sample etc. Examples are given for illustrative purpose.

Keywords: Characterization; Pareto distribution.

Introduction

Economic variation in reported income and true income is studied by error-in-variable model and certain invariance properties of Pareto law. Income reported and true are important subject for ruler (government) for revenue generation. The relation between reported and true (regression) turns out be linear, repotted by Krishnaji and also he asserted that truncated reported income (suitably) agrees to true income in distribution[1]. Nagesh asserted that average under-reporting error for given reported income is linear function of reported income if and only if income follows Pareto law[2].

Henrick[3], Ahsanullah[4,5], Shah[6] and Dimaki[7], used independence of suitable function of order statistics whereas Srivastava[8] used mean and the extreme observation of the sample and characterized Pareto distribution.

Other attempts were made for characterization of exponential and related distributions assuming linear relation of conditional expectation by Beg[9] and Dallas[10], characterization of some types of distributions using recurrence relations between expectations of function of order statistics by Alli[11] and characterization results on exponential and related distributions by Tavangar[12] included characterization of Pareto distribution.

This research note characterizes Pareto distribution by using expectation. Main results of characterization of Pareto distribution is provided in section 2 with proof and section 3 provides examples for illustrative purpose.

The characterization given in section 2 proves the main results and section 3 is gives examples for illustrative purpose.

This research let X have Pareto distribution with probability density function (pdf) as

$$f(x;\theta)=\begin{cases}\dfrac{c\theta^c}{x^{c+1}};0<\theta<x<b,c>0,\\ \\ 0;otherwise\end{cases} \tag{1}$$

Where: $-\infty \le a < b \le \infty$ are known constants, x^{-c-1} is positive absolutely continuous function and $c\theta^c$ is everywhere differentiable function. Since the range of derivative $\left(1/c\theta^c\right)$ being negative and since the range is truncated by θ from left $\left(1/cb^c\right)=0$.

Characterization

In this more general result of characterization through expectation is stated by following theorem.

Theorem 2.1 Let X be a continuous random variable (r.v.) with distribution function $F(X)$, having pdf $f(x;\theta)$. Assume

Research Journal of Mathematical and Statistical Sciences ISSN 2320-6047
Vol. 5(2), 1-4, February (2017) *Res. J. Mathematical and Statistical Sci.*

that $F(X)$ is continuous on the interval (a,b) where $-\infty \le a < b \le \infty$. Let $g(X)$ be differentiable functions of X on the interval (a,b) where $-\infty \le a < b \le \infty$ and more over $g(X)$ be non constant. Then $f(x;\theta)$ is the p.d.f. of Pareto distribution defined in (1) if and only if

$$E\left[g(X)-\frac{X}{c}\frac{d}{dX}g(X)\right]=g(\theta) \quad (2)$$

Proof Given $f(x;\theta)$ defined in (1), if $\phi(X)$ is such that $g(\theta)=E\left[\phi(X)\right]$ where $g(\theta)$ is differentiable function then

$$g(\theta)=\int_{\theta}^{\infty}\phi(x)f(x;\theta)\,dx \quad (3)$$

Differentiating with respect to θ on both sides of (3) and replacing X for θ after simplification one gets

$$\phi(X)=g(X)-\frac{X}{c}\frac{d}{dX}g(X) \quad (4)$$

which establishes necessity of (2). Conversely given (2), let $k(x;\theta)$ be the p.d.f. of r.v. X such that

$$g(\theta)=\int_{\theta}^{\infty}\left[g(x)-\frac{x}{c}\frac{d}{dx}g(x)\right]k(x;\theta)\,dx \quad (5)$$

Since $b^{-c}=0$, the following identity holds

$$g(\theta)=-c\theta^{c}\int_{\theta}^{\infty}\frac{d}{dx}\left(\frac{x^{-c}}{c}g(x)\right)dx \quad (6)$$

which can be rewritten as

$$g(\theta)=-c\theta^{c}\int_{\theta}^{\infty}g(x)\frac{d}{dx}\left(\frac{x^{-c}}{c}\right)+\left(\frac{x^{-c}}{c}\right)\frac{d}{dx}g(x)dx \quad (7)$$

which reduces to

$$g(\theta)=\int_{\theta}^{\infty}\left[g(x)-\frac{x}{c}\frac{d}{dx}g(x)\right]\left\{-c\theta^{c}\frac{d}{dx}\left(\frac{x^{-c}}{c}\right)\right\}dx \quad (8)$$

Using (5) and (8) by uniqueness theorem it follows that p.d.f. of r.v. X

$$k(x;\theta)=-c\theta^{c}\frac{d}{dx}\left(\frac{x^{-c}}{c}\right) \quad (9)$$

Since b^{c} is increasing function for $-\infty \le a < b \le \infty$ and $b^{-c}=0$ is satisfied only when range of X is truncated by θ from left and integrating (9) on the interval (θ,b) on both sides, one gets (9) as

$$1=\int_{\theta}^{\infty}k(x;\theta)\,dx$$

and

$$k(x;\theta)=-c\theta^{c}\frac{d}{dx}\left(\frac{x^{-c}}{c}\right);0<\theta<x<b,c>0 \quad (10)$$

Substituting $\frac{d}{dx}\left(\frac{x^{-c}}{c}\right)$ in (9), $k(x;\theta)$ reduces to $f(x,\theta)$ defined in (1) which establishes sufficiency of (2).

Remark 2.1. Using $\phi(X)$ derived in (4), the $f(x;\theta)$ given in (1) can be determined by

$$M(X)=\frac{\frac{d}{dx}g(x)}{\phi(X)-g(x)} \quad (11)$$

and pdf is given by

$$f(x;\theta)=-\frac{\frac{d}{dx}T(x)}{T(\theta)} \quad (12)$$

where $T(x)$ is decreasing function in the interval (a,b) for $-\infty \le a < b \le \infty$ with $T(b)$ such that it satisfies

$$M(X)=\frac{d}{dx}\log\left[T(x)\right] \quad (13)$$

Examples

Example 3.1 Using method described in the remark characterization of Pareto distribution through survival function $g(\theta)=\overline{F}(t)=\left(\frac{\theta}{t}\right)^{c}$ is illustrated.

$$g(X)=\left(\frac{X}{t}\right)^{c}$$

Research Journal of Mathematical and Statistical Sciences ISSN 2320-6047
Vol. 5(2), 1-4, February (2017) Res. J. Mathematical and Statistical Sci.

$$\phi(X)=g(X)-\frac{X}{c}\frac{d}{dX}g(X)=0$$

$$M(X)=\frac{\frac{d}{dx}g(X)}{\phi(X)-g(X)}=-\frac{X}{c}$$

$$\frac{d}{dX}\log\left[T(X)\right]=-\frac{X}{c}=M(X)$$

$$T(x)=\frac{1}{cX^{c}}$$

$$f(x;\theta)=-\frac{\frac{d}{dx}T(x)}{T(\theta)}=-\frac{\frac{d}{dx}T\left(\frac{1}{cx^{c}}\right)}{T\left(\frac{1}{c\theta^{c}}\right)}=\frac{c\theta^{c}}{x^{c+1}}\quad \text{for } x<\theta.$$

Note that as Hazard function for Pareto distribution being constant it cannot characterize pdf $f(x;\theta)$ given in (1).

Example 3.2 The pdf $f(x;\theta)$ defined in (1) can be characterized through expectation of non constant functions of θ such as

$$g(\theta)=\begin{cases}\frac{c}{c-1}\theta; mean, for-i=1\\ \frac{c}{c-r}\theta^{r}; r^{th} rowmoment, for-i=2\\ e^{\theta}, for-i=3\\ e^{-\theta}, for-i=4\\ \theta(1-p)^{-1/c}, p^{th} quantile, for-i=5\\ 1-\left(\frac{X}{t}\right)^{c}; distribution-function, for-i=6\\ \left(\frac{X}{t}\right)^{c}; reliability-function, for-i=7\end{cases}$$

by using

$$\left[\phi_i(X)-g_i(X)\right]=\begin{cases}-\frac{1}{c-1}X; mean, for-i=1\\ -\frac{r}{c-r}X^{r}; r^{th} rowmoment, for-i=2\\ \frac{X}{t}e^{-X}, for-i=3\\ -\frac{X}{t}e^{X}, for-i=4\\ -\frac{X}{c}(1-p)^{-1/c}, p^{th} quantile, for-i=5\\ \left(\frac{X}{t}\right)^{c}; distribution-function, for-i=6\\ -\left(\frac{X}{t}\right)^{c}; reliability-function, for-i=7\end{cases}$$

and defining $M(X)$ given in (11) and using $T(X)$ as appeared in (13), for (12).

Conclusion

To characterize pdf given in (1) one needs any arbitrary non constant function only.

References

1. Krishnaji N. (1970). Characterization of the Pareto Distribution Through a Model of Underreported Incomes. *Econometrica*, 38(2), 251-255.
2. Nagesh S.R., Michael J.H. and Marcello P. (1974). A Characterization of the Pareto Distribution. *The Annals of Statistics*, 2(3), 599-601.
3. Malik Henrick John (1970). A characterization of the Pareto distribution. *Scandinavian Actuarial Journal*, 3(4), 115-117.
4. Ahsanullah M. and Kabir A.B.M. (1973). A characterization of the Pareto distribution. *Canadian Journal of Statistics*, 1(1-2), 109-112.
5. Ahsanullah M., Lutful Kabir A.B.M. (1974). Characterization of the Pareto distribution. *Communication in statistics*, 3(10), 953-957.
6. Shah S.M. and Kabe D.G. (1981). Characterizations of Exponential, Pareto, Power Function, BURR and Logistic Distributions by Order Statistics. *Biometrical Journal*, 23(2), 141-146.

Research Journal of Mathematical and Statistical Sciences ______ ISSN 2320-6047
Vol. **5(2)**, 1-4, February **(2017)** *Res. J. Mathematical and Statistical Sci.*

7. Dimaki C. and Xekalaki Evdokia (1993). Characterizations of the *Pareto distribution* based on order statistics. Stability Problems for Stochastic Models, Lecture Notes in Mathematics, 1546, 1-16.
8. Srivastava M.S. (1965). Characterizations of Pareto's Distributions and $(k+1)x^k/\wp^{k+1}$. *Ann. Math. Statist.*, 36, 361-362.
9. Beg M.L. and Kirmani S.N.U.A. (1974). On a characterization of exponential and related distributions. Austral. J. Statist., 16 (3), 163-166.
10. Dallas A.C. (1976). Characterization Pareto and power distribution. *Ann. Ins. Statist. Math*, 28(1), 491-497.
11. Ali M.A. and Khan A.H. (1998). Characterization of Some Types of Distributions *Information and Management Sciences*, 9(2), 1-9.
12. Tavangar M. and Asadi M. (2011). Some new characterization results on exponential and related distributions *Bulletin of the Iranian Mathematical Society*, 36(1), 257-272.

Journal of Statistical Science and Application, August 2016, Vol. 4, No. 07-08, 196-203
doi: 10.17265/2328-224X/2016.0708.004

DAVID PUBLISHING

Characterization of Pareto Distribution Through Expectation of Function of Order Statistics

Bhatt Milind. B
Department of Statistics
Sardar Patel University
Vallabhvidyanagar-388129
Dist. Anand, Gujarat
INDIA
bhattmilind_b@yahoo.com

For characterization of Pareto distribution one needs any arbitrary non constant function only by approach of identity of distribution and equality of expectation of function of random variable in place of approaches such as relation (linear) in (economic variation) reported and true income, independency of suitable function of order statistics, mean and the extreme observation of the sample etc. Examples are given for illustrative purpose.

Keyword : Characterization; Pareto distribution

Introduction

Certain skew pattern appear in socioeconomic quantities such stock price fluctuation, personal income, economic variation in reported income and under-reporting error [See. Krishnaji (1970), Nagesh (1974)] have certain invariant properties for which Pareto distribution foun most duitable. Amongst many other Pareto distribution used to study skew pattern. Pareto distribution also used to study empiric phenomena such as occurrence of natural resources, error clustering in communication circuit, size of firm, city, population and reliability theory.

Independence of suitable function of order statistics was used for characterization of Pareto distribution by Henrick (1970), Ahsanullah (1973, 1974), Shah (1981) and Dimaki (1993) where as Srivastava (1976) used mean and the extreme observation of the sample.

Other attempts were made for characterization of exponential and related distributions assuming linear relation of conditional expectation by Beg (1974) and Dallas (1976), characterization of some types of distributions using recurrence relations between expectations of function of order statistics by Alli (1998) and characterization results on exponential and related distributions by Tavangar (2010) included characterization of Pareto distribution.

This research note provides the characterization based on identity of distribution and equality of expectation of function of order statistics for Pareto distribution with the probability density function (p.d.f.).

Corresponding author: Bhatt Milind. B, Department of Statistics, Sardar Patel University, Vallabhvidyanagar-388129, Dist. Anand, Gujarat, INDIA. E-mail: bhattmilind_b@yahoo.com.

This work was supported by UGC Major Research Project : F.No.42-39/2013(SR) dated 12-3-2013

This research note provides characterization of Pareto distribution with probability density function (pdf)

$$f(x,\theta)=\begin{cases}\frac{c\theta^c}{x^{c+1}}\ ;\ \theta<x<\infty;\ c>0, \\ \\ 0\ ;\ otherwise,\end{cases} \quad \ldots \tag{1.1}$$

where $-\infty \le a < b \le \infty$ are known constants and $(1/x)^{c+1}$ is positive absolutely continuous function and $c\theta^c$ is everywhere differentiable function is characterized.

Note that c is income concentration use as measure of inequalities in income distribution and θ is minimum level of income.

The aim of the present research note is to give path braking new characterization for Pareto distribution defined in (1.1) through expectation of function of order statistics, using identity and equality of expectation. Characterization theorem provrd in section 2 with method for characterization as remark and section 3 devoted to applications for illustrative purpose.

Characterization Theorem

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from distribution function f and let $X_{1:n} < X_{2:n} < \ldots < X_{nn:}$ be the set of corresponding order statistics. Assume that f is continuous on the interval (a,b) where $-\infty \le a < b \le \infty$. Let $g(X_{1:n})$ and $\phi(X_{1:n})$ be be two distinct differentiable and intregrable functions of first order statistic; $X_{1:n}$ on the interval (a,b) where $-\infty \le a < b \le \infty$. and moreover \$ $g(X_{1:n})$ be non-constant function of $X_{1:n}$. Then

$$E\left[g(X_{1:n})-\left(\frac{X_{1:n}}{\mathrm{nc}}\right)\frac{d}{dX_{1:n}}g(X_{1:n})\right]=g(\theta)\ \ldots \tag{2.1}$$

is the necessary and sufficient condition for pdf f of F to be $f(x,\theta)$ defined in (1.1).

Proof. Given $f(x,\theta)$ defined in (1.1), for necessity of (2.1) if $\phi(X_{1:n})$ is such that $g(\theta)=E\phi(X_{1:n})$ where $g(\theta)$ is differentiable function then using $f(X_{1:n};\theta)$ pdf of first order statistic; $X_{1:n}$ one gets

$$\mathrm{g}(\theta)=\int_\theta^b \phi(\mathrm{x}_{1:n})\,\mathrm{f}(\mathrm{x}_{n:n},\theta)\,\mathrm{dx}_{n:n}\ \ldots \tag{2.2}$$

. Differentiating (2.2) with respect to θ on both sides and replacing $X_{1:n}$ for θ and simplifying one gets

$$\phi(\mathrm{x}_{1:n})=g(X_{1:n})-\left(\frac{X_{1:n}}{\mathrm{nc}}\right)\frac{d}{dX_{1:n}}g(X_{1:n})\ \ldots \tag{2.3}$$

which establishes necessity of (2.1). Conversely given (2.1), let $k(x_{1:n};\theta)$ be the pdf of first order statistic; $X_{1:n}$ such that

$$\mathrm{g}(\theta)=\int_\theta^b \phi(\mathrm{x}_{1:n})\,\mathrm{k}(\mathrm{x}_{1:n},\theta)\,\mathrm{dx}_{1:n}\ \ldots \tag{2.4}$$

Since $(ncx_{n:n}^{nc})$; $(n\ge 1)$ is increasing integrable and differentiable function on the interval (a,b) with $\left(\frac{1}{ncb^{nc}}\right)=0$ the following identity holds

$$g(\theta)\equiv -nc\theta^{nc}\int_\theta^b\left[\frac{d}{dx_{1:n}}\left\{g(x_{1:n})\left(\frac{1}{ncx_{1:n}^{nc}}\right)\right\}\right]dx_{1:n}.\ \ldots \tag{2.5}$$

Differentiating integrand $g(x_{1:n})\left(\frac{1}{ncx_{1:n}^{nc}}\right)$ with respect to $x_{1:n}$ and simplifying after taking $\frac{d}{dX_{1:n}}\left(\frac{1}{ncx_{1:n}^{nc}}\right)$

as one factor one gets (2.5) as

$$g(\theta) \equiv \int_\theta^b \left[g(X_{1:n}) + \frac{\left(\frac{X_{1:n}}{nc}\right)}{\frac{d}{dX_{1:n}}\left(\frac{1}{ncx_{1:n}^{nc}}\right)} \frac{d}{dX_{1:n}} g(X_{1:n}) \right] \left\{ -nc\theta^{nc} \frac{d}{dX_{1:n}} \left(\frac{1}{ncx_{1:n}^{nc}}\right) \right\} dx_{1:n} \; \dots \qquad (2.6)$$

and substituting derivative of $\left(\frac{1}{ncx_{1:n}^{nc}}\right)$ in (2.6) one gets

$$g(\theta) \equiv \int_\theta^b \phi(x_{1:n}) \frac{nc\theta^{nc}}{x_{1:n}^{nc+1}} dx_{1:n} \; \dots \qquad (2.7)$$

where $\phi(x_{1:n})$ is as derived in (2.3). By uniqueness theorem from (2.4) and (2.7)

$$k(x_{1:n}, \theta) = \frac{nc\theta^{nc}}{x_{1:n}^{nc+1}} \; \dots \qquad (2.8)$$

Since $(ncx_{1:n}^{nc})$; $(n \geq 1)$ is increasing integrable and differentiable function on the interval (a, b) with $\left(\frac{1}{ncb^{nc}}\right) = 0$ and since $(ncx_{1:n}^{nc})$ is increasing function for $-\infty \leq a < b \leq \infty$ with $\left(\frac{1}{ncb^{nc}}\right) = 0$. is satisfy only when range of $X_{1:n}$ is truncated by θ from left and integrating (2.8) on the interval (θ, b) on both sides, one gets

$$1 = \int_\theta^b k(x_{1:n}, \theta)\, dx_{1:n}$$

For $n = 1$, $[k(x_{1:n}, \theta)]_{n=1}$ reduces to $f(x, \theta)$ defined in (1.1). Hence sufficiency of (2.1) is established.

Remark 2.1. Using $\phi(X)$ given in (2.2) one can determine $f(x, \theta)$ by

$$M(X_{1:n}) = \frac{\frac{d}{dX_{1:n}} g(X_{1:n})}{\phi(X_{1:n}) - g(X_{1:n})} \; \dots \qquad (2.9)$$

and pdf is given by

$$f(x, \theta) = \left[-\frac{\frac{d}{dx_{1:n}} T(x_{1:n})}{T(\theta)} \right]_{n=1}, \; a < \theta < x < b \; \dots \qquad (2.10)$$

where $T(X)$ is decreasing function for $-\infty \leq a < b \leq \infty$ with $T(b) = 0$ such that it satisfies

$$M(X_{1:n}) = \frac{d}{dX_{1:n}} [\log T(X_{1:n})]. \; \dots \qquad (2.11)$$

Remark 2.1. The theorem 2.1 for function of first order statistics with remark 2.1also holds for random variable X when $(n = 1)$.

3. Examples. Using method describe in remark 2.1 Pareto distribution through expectation of non-constant function of order statistics is characterized as illustrative example and significant of unified approach of characterization result.

Example 3.1 Characterization of Pareto distribution through the Minimum Variance Unbiased (UMVU) estimator $\widehat{e^\theta}$ of e^θ is given.

$$\widehat{e^\theta} = e^{X_{1:n}} \left(1 - \frac{X_{1:n}}{nc}\right) = g(X_{1:n})$$

Using (2.3) one gets

$$\phi(x_{1:n}) = g(X_{1:n}) - \left(\frac{X_{1:n}}{nc}\right)\frac{d}{dX_{1:n}}g(X_{1:n}) = e^{X_{1:n}}\left(1 - 2\frac{X_{1:n}}{nc} + \left(\frac{X_{1:n}}{nc}\right)^2 + \frac{X_{1:n}}{(nc)^2}\right)$$

and (2.9) of remark 2.1

$$M(X_{1:n}) = \frac{\frac{d}{dX_{1:n}}g(X_{1:n})}{\phi(X_{1:n}) - g(X_{1:n})} = -\frac{nc}{X_{1:n}}$$

with

$$\frac{d}{dx_{1:n}}\left[\log\left(\frac{1}{ncx_{1:n}^{nc}}\right)\right] = M(X_{1:n})$$

then

$$T(x_{1:n}) = -\frac{1}{x_{1:n}^{nc+1}}$$

and

$$f(x,\theta) = \left[-\frac{\frac{d}{dx_{1:n}}T(x_{1:n})}{T(\theta)}\right]_{n=1} = \frac{c\theta^c}{x^{c+1}},\ a < \theta < x < b$$

Example 3.2 Characterization of Pareto distribution through the uniformly minimum variance unbiased (UMVU) estimator $\hat{g}(\theta)$ and maximum likelihood estimator (MLE) $\bar{g}(\theta)$ of $g(\theta)$ such as $\mu_1'(\theta)$; mean, $\mu_1'(\theta)$; r[th] moment $e^{\theta}, e^{-\theta}$, $Q_p(\theta)$; p[th] quantile, $F(t)$; distribution function and $\bar{F}(t)$; reliability function is given. For the (UMVU) estimator

$$\hat{g}_i(\theta) = \begin{cases} \hat{\mu}_1'(\theta) = \frac{c}{c-1}\left[1 - \frac{1}{nc}\right]X_{1:n};\ for\ i = 1 \\ \hat{\mu}_r'(\theta) = \frac{c}{c-r}\left[1 - \frac{r}{nc}\right]X_{1:n}^r;\ for\ i = 2 \\ \widehat{e^{\theta}} = e^{X_{1:n}}\left(1 - \frac{X_{1:n}}{nc}\right);\ for\ i = 3 \\ \widehat{e^{-\theta}} = e^{-X_{1:n}}\left(1 + \frac{X_{1:n}}{nc}\right);\ for\ i = 4 \\ \hat{Q}_p(\theta) = X_{1:n}(1-p)^{-\frac{1}{c}}\left\{1 - \frac{1}{nc}\right\};\ for\ i = 5 \\ \hat{F}(t) = 1 - \left(\frac{X_{1:n}}{t}\right)^c\left\{1 - \frac{1}{n}\right\};\ for\ i = 6 \\ \hat{\bar{F}}(t) = \left(\frac{X_{1:n}}{t}\right)^c\left\{1 - \frac{1}{nc}\right\};\ for\ i = 7 \end{cases}$$

and MLE

$$\tilde{g}_i(\theta) = \begin{cases} \tilde{\mu}_1'(\theta) = \dfrac{c}{c-1} X_{1:n};\ for\ i = 1 \\ \tilde{\mu}_r'(\theta) = \dfrac{c}{c-r} X_{1:n}^r;\ for\ i = 2 \\ \widetilde{e^{\theta}} = e^{X_{1:n}};\ for\ i = 3 \\ \widetilde{e^{-\theta}} = e^{-X_{1:n}};\ for\ i = 4 \\ \tilde{Q}_p(\theta) = X_{1:n}(1-p)^{-\frac{1}{c}};\ for\ i = 5 \\ \tilde{F}(t) = 1 - \left(\dfrac{X_{1:n}}{t}\right)^c;\ for\ i = 6 \\ \tilde{\bar{F}}(t) = \left(\dfrac{X_{1:n}}{t}\right)^c;\ for\ i = 7 \end{cases}$$

One gets

$$[\phi_i(x_{1:n}) - \tilde{g}_i(\theta)] = \begin{cases} \left(-\dfrac{1}{nc}\right)\dfrac{c}{c-1}\left[\dfrac{nc-1}{nc}\right] X_{1:n};\ for\ i = 1 \\ -\left(\dfrac{r}{nc}\right)\left(\dfrac{c}{c-r}\right)\left[1 - \dfrac{r}{nc}\right] X_{1:n}^r;\ for\ i = 2 \\ -\dfrac{X_{1:n}e^{X_{1:n}}}{nc}\left\{1 - \dfrac{X_{1:n}}{nc} - \dfrac{1}{nc}\right\};\ for\ i = 3 \\ \dfrac{X_{1:n}e^{-X_{1:n}}}{nc}\left\{1 + \dfrac{X_{1:n}}{nc} - \dfrac{1}{nc}\right\};\ for\ i = 4 \\ -\dfrac{X_{1:n}}{nc}(1-p)^{-\frac{1}{c}}\left\{1 - \dfrac{1}{nc}\right\};\ for\ i = 5 \\ \dfrac{1}{n}\left\{1 - \dfrac{1}{n}\right\}\left(\dfrac{X_{1:n}}{t}\right)^c;\ for\ i = 6 \\ -\dfrac{1}{n}\left(\dfrac{X_{1:n}}{t}\right)^c\left\{1 - \dfrac{1}{nc}\right\};\ for\ i = 7 \end{cases}$$

and

$$[\phi_i(x_{1:n}) - \bar{g}_i(\theta)] = \begin{cases} -\frac{c}{c-1}\left(\frac{X_{1:n}}{nc}\right); for\ i = 1 \\ -\left(\frac{c}{c-r}\right)\frac{rX_{1:n}^r}{nc}; for\ i = 2 \\ -\frac{X_{1:n}e^{X_{1:n}}}{nc}; for\ i = 3 \\ \frac{X_{1:n}e^{-X_{1:n}}}{nc}; for\ i = 4 \\ -\frac{X_{1:n}(1-p)^{-\frac{1}{c}}}{nc}; for\ i = 5 \\ \frac{1}{n}\left(\frac{X_{1:n}}{t}\right)^c; for\ i = 6 \\ -\frac{1}{n}\left(\frac{X_{1:n}}{t}\right)^c; for\ i = 7 \end{cases}$$

respectively.

Then by defining $M(X_{1:n})$ given in (2.9) and substituting $T(x_{1:n})$ as appeared in (2.11) for (2.10), $f(x,\theta)$ is characterized.

Example 3.3 In context of remark 2.2 characterization of Pareto distribution through p[th] quantile ; $\tilde{Q}_p(\theta)$ is given.

$$\tilde{Q}_p(\theta) = \theta(1-p)^{-\frac{1}{c}}$$

Therefore

$$g(\theta) = X(1-p)^{-\frac{1}{c}}$$

and from (2.3)

$$\phi(X) = g(X) - \left(\frac{X}{c}\right)\frac{d}{dX}g(X) = \left(\frac{c-1}{c}\right)(1-p)^{-\frac{1}{c}}X$$

and (2.9) of remark 2.1

$$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X) - g(X)} = -\frac{c}{X}$$

with

$$\frac{d}{dx_{1:n}}\left[\log\left(\frac{1}{cx^c}\right)\right] = M(X_{1:n})$$

then

$$T(X) = -\frac{1}{cx^c}$$

and

$$f(x,\theta) = -\frac{\frac{d}{dX}T(X)}{T(\theta)} = \frac{c\theta^{c}}{x^{c+1}},\ a < \theta < x < b.$$

Example 3.4 Using remark 2.2 the pdf $f(x,\theta)$ defined in (1.1) can be characterized through non constant functions of θ such as

$$g_i(\theta) = \begin{cases} \mu_1'(\theta) = \dfrac{c}{c-1}\theta;\ for\ i = 1 \\ \mu_r'(\theta) = \dfrac{c}{c-r}\theta^{r};\ for\ i = 2 \\ e^{\theta};\ for\ i = 3 \\ e^{-\theta};\ for\ i = 4 \\ Q_p(\theta) = \theta(1-p)^{-\frac{1}{c}};\ for\ i = 5 \\ F(t) = 1 - \left(\dfrac{\theta}{t}\right)^{c};\ for\ i = 6 \\ \bar{F}(t) = \left(\dfrac{\theta}{t}\right)^{c};\ for\ i = 7 \end{cases}$$

by using

$$[\phi_i(x_{1:n}) - g_i(\theta)] = \begin{cases} -\dfrac{X}{c-1};\ for\ i = 1 \\ -\dfrac{r\,X^{r}}{c-r};\ for\ i = 2 \\ -\dfrac{Xe^{X}}{c};\ for\ i = 3 \\ \dfrac{Xe^{-X}}{c};\ for\ i = 4 \\ -\dfrac{X}{C}(1-p)^{-\frac{1}{c}};\ for\ i = 5 \\ \left(\dfrac{X}{t}\right)^{c};\ for\ i = 6 \\ -\left(\dfrac{X}{t}\right)^{c};\ for\ i = 7 \end{cases}$$

Then by defining $M(X_{1:n})$ given in (2.9) and substituting $T(x_{1:n})$ as appeared in (2.11) for (2.10), $f(x,\theta)$ is characterized.

Reference

Ahsanullah, (1973). A characterization of the Pareto distribution. Canadian Journal of Statistics, Volume 1, Issue 1-2, pages 109–112, 1973

Ahsanullah (1974). Characterization of the Pareto distribution. communication in statistics, 3(10), 953-957, 1974.

Ali, M. A. and Khan, A. H. (1998). Characterization of Some Types of Distributions, Information and Management Sciences, Vol. 9, No. 2, June, 1998.

Beg, M. L and Kirmani, S. N. U. A, (1974). On a characterization of exponential and related distributions, Austral. J. Statist., 16 (3), 1974, 163-166.

Dallas, A. C. (1976). Characterization Pareto and power distribution, Ann. Ins. Statist. Math, 28, (1976), Part A, pp. 491 – 497

Dimaki,C.and Evdokia Xekalaki. (1993). Characterizations of the *Pareto distribution* based on order statistics. Stability Problems for Stochastic Models, Lecture Notes in Mathematics Volume 1546, pp 1-16.

Henrick John Malik, (1970). A characterization of the Pareto distribution. Scandinavian Actuarial Journal, Volume 1970, Issue 3-4, 1970.

Krishnaji, N, (1970). Characterization of the Pareto Distribution Through a Model of Underreported Incomes. Vol. 38, No. 2, Mar., 251-255, 1970.

Nagesh S. R., Michael J. H. and Marcello P. (1974). A Characterization of the Pareto Distribution. Institute of Mathematical Statistics Vol. 2, No.3, Page 599 of 599-601, 1974.

Shah, S. M. and Kabe, D. G, (1981). Characterizations of Exponential, Pareto, Power Function, BURR and Logistic Distributions by Order Statistics. Biometrical Journal, volume 23, Issue 2, pages141-146.

Srivastava, M. S. (1965). Characterizations of Pareto's Distributions and $(k+1)x^k/\theta^{k+1}$. Ann. Math. Statist.,36, 361-362.

Tavangar, M. and Asadi, M. (2019). some new characterization results on exponential and related distributions, Bulletin of the Iranian Mathematical Society Vol. 36 No. 1 (2010), pp 257-272.

Open Journal of Statistics, **2013, 3, 441-443**
Published Online December 2013 (http://www.scirp.org/journal/ojs)
http://dx.doi.org/10.4236/ojs.2013.36052

Scientific Research

Characterization of Power-Function Distribution through Expectation*

Milind Bhanuprasad Bhatt
Department of Statistics, Sardar Patel University, Anand, India
Email: bhattmilind_b@yahoo.com

Received October 12, 2013; revised November 12, 2013; accepted November 19, 2013

ABSTRACT

For the characterization of the power function distribution, one needs any arbitrary non constant function only in place of independence of suitable function of order statistics, linear relation of conditional expectation, recurrence relations between expectations of function of order statistics, distributional properties of exponential distribution, record valves, lower record statistics, product of order statistics and Lorenz curve, etc. available in the literature. The goal of this research is not to give a different path-breaking approach for the characterization of power function distribution through the expectation of non constant function of random variable and provide a method to characterize the power function distribution as remark. Examples are given for the illustrative purpose.

Keywords: Characterization; Power Function Distribution

1. Introduction

Several characterizations of power function distribution have been made notably by Fisz [1], Basu [2], Govindarajulu [3] and Dallas [4] using independence of suitable function of order statistics and distributional properties of transformation of exponential variable.

Other attempts were made for the characterization of exponential and related distributions assuming linear relation of conditional expectation by Beg [5], characterization based on record valves by Nagraja [6], characterization of some types of distributions using recurrence relations between expectations of function of order statistics by Alli [7], characterization results on exponential and related distributions by Tavangar [8], and characterization continuous distributions through lower record statistics by Faizan [9] included the characterization of power function distribution.

Direct characterization for power function distribution has been given in Arslan [10] who used the product of order statistics [contraction is a particular case of product of order statistics which has interesting applications such as in economic modeling and reliability see Alamatsaz [11], Kotz [12] and Alzaid [13]] where as Moothathu [14] used Lorenz curve. [Graph of fraction of total income owned by lowest p[th] fraction of the population is Lorenz curve of distribution of income [15].

This research note provides the characterization based on identity of distribution and equality of expectation of function of random variable for power-function distribution with the probability density function (p.d.f.)

$$f(x;\theta) = \begin{cases} c\theta^{-c}x^{c-1}; a<x<\theta<b; \theta=k^{-1}, k>0, c>0 \\ 0; \quad \text{otherwise} \end{cases} \quad (1.1)$$

where $-\infty \le a < b \le \infty$ are known constants, x^{c-1} is positive absolutely continuous function and c/θ^c is everywhere differentiable function. Since derivative of x^c/c being positive and since range is truncated by θ from right $(a^c/c)=0$.

The aim of the present research note is to give the new characterization through the expectation of function $\phi(x)$ for the power function distribution. Examples are given for the illustrative purpose.

*This work is supported by UGC Major Research Project No: F.No. 42-39/2013(SR), dated 12-3-2013.

2. Characterization

Theorem 2.1 Let X be a random variable with distribution function F. Assume that F is continuous on the interval, (a,b) where $-\infty \le a < b \le \infty$. Let $\phi(x)$ and $g(X)$ be two distinct differentiable and integrable functions of X on the interval (a,b) where $-\infty \le a < b \le \infty$ and moreover $g(X)$ be non constant. Then $f(x;\theta)$ is the p.d.f. of power function distribution defined in (1.1) if and only if

$$E\left[g(x)+\frac{X}{c}\frac{\mathrm{d}}{\mathrm{d}X}g(x)\right]=g(\theta) \tag{2.1}$$

Proof Given $f(x;\theta)$ defined in (1.1), if $\phi(x)$ is such that $g(\theta)=E\phi(X)$ where $g(\theta)$ is differentiable function then

$$g(\theta)=\int_a^\theta \phi(x)f(x;\theta)\mathrm{d}x \tag{2.2}$$

Differentiating (2.2) with respect to θ on both sides and replacing X for θ and simplifying one gets

$$\phi(x)=g(x)+\frac{X}{c}\frac{\mathrm{d}}{\mathrm{d}X}g(x) \tag{2.3}$$

which establishes necessity of (2.1). Conversely given (2.1), let $k(x;\theta)$ be such that

$$g(\theta)=\int_a^\theta\left[g(x)+\frac{X}{c}\frac{\mathrm{d}}{\mathrm{d}X}g(x)\right]k(x;\theta)\mathrm{d}x \tag{2.4}$$

Since $(a^c/c)=0$ the following identity holds:

$$g(\theta)=\frac{c}{\theta^c}\int_a^\theta\left[\frac{\mathrm{d}}{\mathrm{d}x}g(x)\left(\frac{x^c}{c}\right)\mathrm{d}x\right] \tag{2.5}$$

Differentiating integrand of (2.5) and tacking $\frac{\mathrm{d}}{\mathrm{d}x}(x^c/c)$ as one factor one gets (2.5) as

$$g(\theta)=\int_a^\theta\phi(x)\left\{c\theta^{-c}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^c}{c}\right)\right\}\mathrm{d}x \tag{2.6}$$

where $\phi(x)$ is function of X derived in (2.3). From (2.4) and (2.6) by uniqueness theorem

$$k(x;\theta)=\frac{c}{\theta^c}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^c}{c}\right) \tag{2.7}$$

Since x^c/c is decreasing function with $(a^c/c)=0$ and since $\theta=K^{-1}, K>0$, integrating (2.7) on both sides one gets

$$1=\int_a^\theta k(x;\theta)\mathrm{d}x \tag{2.8}$$

Substituting $\frac{\mathrm{d}}{\mathrm{d}x}(x^c/c)$ in (2.7), $k(x;\theta)$ reduces to $f(x;\theta)$ defined in (1.1), which establishes sufficiency of (2.1).

Note: Author does not claim the relations between f and g in the preceding analysis.

Remark 2.1 Using $\phi(x)$ derived in (2.3), $f(x;\theta)$ given in (1.1) can be determined by

$$M(X)=\frac{\frac{\mathrm{d}}{\mathrm{d}X}g(X)}{\phi(X)-g(X)} \tag{2.9}$$

and p.d.f. is given by

$$f(x;\theta)=\frac{\frac{\mathrm{d}}{\mathrm{d}x}U(x)}{U(\theta)} \tag{2.10}$$

where $U(X)$ is increasing function in the interval (a,b) for $-\infty \le a < b \le \infty$ with $U(a)=0$ such that it satisfies

$$M(X)=\frac{\mathrm{d}}{\mathrm{d}X}\log U(X)$$

3. Illustrative Examples

Example 1 Using method described in the remark characterization of power function distribution through survival function quantile; $Q_p(\theta)=p^{\frac{1}{c}}\theta$ is illustrated.

$$g(X)=p^{1/c}X$$

$$\phi(x)=g(x)+\frac{X}{c}\frac{\mathrm{d}}{\mathrm{d}X}g(x)=\frac{c+1}{c}Xp^{1/c}$$

$$M(X)=\frac{\frac{\mathrm{d}}{\mathrm{d}X}g(X)}{\phi(X)-g(X)}=\frac{c}{X}$$

$$\frac{\mathrm{d}}{\mathrm{d}X}\log\left(\frac{X}{c}\right)=\frac{c}{X}=M(X)$$

$$U(x)=\frac{X^c}{c}$$

$$f(x;\theta)=\frac{\frac{\mathrm{d}}{\mathrm{d}x}U(x)}{U(\theta)}=\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^c}{c}\right)}{\frac{\theta^c}{c}}$$

$$=c\theta^{-c}x^{c-1}, x>\theta$$

Example 2 The p.d.f. $f(x;\theta)$ defined in (1.1) can be characterized through non constant functions of θ such as

$$g_i(\theta)=\begin{cases}\dfrac{c}{c+1}\theta;\text{mean}\\ \dfrac{c}{c+r}\theta^r;r^{th}\text{raw-moment}\\ e^{\theta}\\ e^{-\theta}\\ \theta p^{-1/c};p^{th}\text{quantile}\\ \left(\dfrac{t}{\theta}\right)^c;\text{distribution-function}\\ 1-\left(\dfrac{t}{\theta}\right)^c;\text{reliability-function}\\ \dfrac{\left(\dfrac{t}{\theta}\right)}{\left(\dfrac{\theta}{t}\right)^c-1};\text{hazard-function}\end{cases}$$

by using

$$\left[\phi_i(X)-g_i(X)\right]=\begin{cases}\\ \dfrac{X}{c+1};\text{mean}\\ \dfrac{X}{c}e^{X}\\ -\dfrac{X}{c}e^{-X}\\ \dfrac{X}{c}p^{1/c};p^{th}\text{quantile}\\ -\left(\dfrac{t}{X}\right)^c;\text{distribution-function}\\ \left(\dfrac{t}{X}\right)^c;\text{reliability-function}\\ \dfrac{\left(\dfrac{c}{t}\right)\left(\dfrac{t}{X}\right)^c}{\left[1-\left(\dfrac{t}{X}\right)^c\right]^2};\text{hazard-function}\end{cases}$$

and defining $M(X)$ given in (2.9) and using $U(X)$ as appeared in (2.11) for (2.10).

4. Conclusion

To characterize the p.d.f. defined in (1.1), one needs any arbitrary non constant function of X which should be differentiable and integrable only.

REFERENCES

[1] M. Fisz, "Characterization of Some Probability Distribution," *Skand. Aktuarietidskr*, Vol. 41, No. 1-2, 1958, pp. 65-67.

[2] A. P. Basu, "On Characterizing the Exponential Distribution by Order Statistics," *Annals of the Institute of Statistical Mathematics*, Vol. 17, No. 1, 1965, pp. 93-96. http://dx.doi.org/10.1007/BF02868158

[3] Z. Govindarajulu, "Characterization of Exponential and Power Distribution," *Skand. Aktuarietidskr*, Vol. 41, 1966, pp. 132-136.

[4] A. C. Dallas, "Characterization Pareto and Power Distribution," *The Annals of Mathematical Statistics, Part A*, Vol. 28, No. 1, 1976, pp. 491-497.

[5] M. L. Beg and S. N. U. A. Kirmani, "On a Characterization of Exponential and Related Distributions," *The Australian Journal of Statistics*, Vol. 16, No. 3, 1974, pp. 163-166. http://dx.doi.org/10.1111/j.1467-842X.1974.tb00933.x

[6] H. N. Nagaraja, "On a Characterization Based on Record Valves," *The Australian Journal of Statistics*, Vol. 19, No. 1, pp. 70-73. http://dx.doi.org/10.1111/j.1467-842X.1977.tb01078.x

[7] M. A. Ali and A. H. Khan, "Characterization of Some Types of Distributions," *Information and Management Sciences*, Vol. 9, No. 2, 1998, pp. 1-9.

[8] M. Tavangar and M. Asadi, "Some New Characterization Results on Exponential and Related Distributions," *Bulletin of the Iranian Mathematical Society*, Vol. 36, No. 1, 2010, pp. 257-272.

[9] M. Faizan and M. I. Khan, "A Characterization of Continuous Distributions through Lower Record Statistics," *Forum*, Vol. 4, 2011, pp. 39-43.

[10] G. Arslan, "Characterization Based on Product of Order Statistics," math.ST., 2011. arXiv 1110.2879v

[11] M. H. Alamatsaz, "An Integer-Valued PTH-Order Autoregressive Structure," (INAR(p)) Process," *Journal of Applied Probability*, Vol. 27, No. 2, 1990, pp. 314-324.

[12] S. Kotz and F.W. Steutel, "Note on a Characterization of Exponential Distributions," *Statistics & Probability Letters*, Vol. 6, No. 3, 1988, pp. 201-203. http://dx.doi.org/10.1016/0167-7152(88)90120-4

[13] A. A. Alzaid and M. A. Al-Osh, "An Integer-Valued PTH-Order Autoregressive Structure," (INAR(p)) Process," *Journal of Applied Probability*, Vol. 27, No. 2, 1990, pp. 314-324. http://dx.doi.org/10.2307/3214650

[14] T. S. K. Moothathu, "Characterization of Power Function Distribution through Property of Lorenz Curve," *Sankhy, Journal of Statistics, Series B*, Vol. 48, Pt. 2, 1986, pp. 262-265.

[15] M. G. Kendall and A. Stuart, "The Advanced Theory of Statistics," C. Griffin, London, 1958.

DOI: 10.13189/ms.2015.030602 http://www.hrpub.org

Characterization of Power Function Distribution through Expectation of Function of Order Statistics

Bhatt Milind B.

Department of Statistics, Sardar Patel University, Vallabh Vidyanagar-388120, Dist: Anand, State: Gujarat, India

Abstract Independence of suitable function of order statistics, linear relation of conditional expectation, recurrence relations between expectations of function of order statistics, distributional properties of exponential distribution, record valves, lower record statistics, product of order statistics and Lorenz curve etc.. are various approaches available in the literature for the characterization of the power function distribution. In this research note different path breaking approach for the characterization of power function distribution through the expectation of function of order statistics is given and provides a method to characterize the power function distribution which needs any arbitrary non constant function only.

Keywords Characterization, Power function distribution , Probability Density Function.

1 Introduction

Notable attempt to characterized Power function distribution through independence of suitable function of order statistics and distributional properties of transformation of exponential are Basu [1], Govindarajulu [2], Desu [3] and Dallas [4] where as of exponential and related distributions assuming linear relation of conditional expectation by Beg [5], characterization based on record values by Nagraja [6], characterization of some types of distributions using recurrence relations between expectations of function of order statistics by Alli [7], characterization results on exponential and related distributions by Tavangar [8] and characterization of continuous distributions through lower record statistics by Faizan [9] included the characterization of power function distribution as special case.

Direct characterization for power function distribution has been given in Fisz [10] who use independence properties of order statistics where as Arslan [11] used product of order statistic. [contraction is a particular case of product of order statistics which has interesting pplications such as in economic modeling and reliability see Alamatsaz [12], Kotz [13] and Alzaid [14]] where as Moothathu [15] used Lorenz curve. [Graph of fraction of total income owned by lowest pth fraction of the population is Lorenz curve of distribution of income of distribution of income.][See. Kendall and Stuart [16]].

This research note provides the characterization based on identity of distribution and equality of expectation of function of order statistics for power-function distribution with the probability density function (p.d.f.)

$$f(x;\theta) = \begin{cases} c\theta^{-c}x^{c-1}; & a < x < \theta < b; \theta = k^{-1}, k > 0, c > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $-\infty < a < b < \infty$ are known constants, x^{c-1} is positive absolutely continuous function and $(\frac{x}{\theta})^c$ is everywhere differentiable function. Since derivative of x^{c-1} being positive and since range is truncated by θ from right for $f(x;\theta)$ defined in (1.1), $\frac{a^c}{c} = 0$.

The aim of the present research note is to give the new characterization through the expectation of function of order statistics, using identity and equality of expectation. Characterization theorem derived in section 2 with method for characterization as remark and section 3 devoted to applications for illustrative purpose.

2 Characterization theorem

Theorem

Let $X_1, X_{2,}, X_n$ be a random sample of size n from distribution function F. Let $X_{1:n} < X_{2:n}, \dots, < X_{n:n}$ be the set of corresponding order statistics. Assume that F is continuous on the interval (a,b) where $-\infty < a < b < \infty$. Let $g(X_{n:n})$ and $\phi(X_{n:n})$ be be two distinct differentiable and intregrable functions of n^{th} order statistic; $X_{n:n}$ on the interval (a, b) where $-\infty < a < b < \infty$ and moreover $g(X_{n:n})$ be non-constant function of $X_{n:n}$. Then

$$E\Big[g(X_{n:n}) + \Big(\frac{X_{n:n}}{nc}\Big)\frac{d}{dX_{n:n}}g(X_{n:n})\Big] = g(\theta). \tag{2}$$

is the necessary and sufficient condition for pdf $f(x;\theta)$ of F to be $f(x;\theta)$ defined in (1).

Proof

Given $f(x;\theta)$ defined in (1), for necessity of (2) if $\phi(X_{n:n})$ is such that $g(\theta) = E[\phi(X_{n:n})]$ where $g(\theta)$ is differentiable function then using $f(x_{n:n};\theta)$; pdf of n^{th} order statistic one gets,

$$g(\theta) = \int_a^\theta \phi(x_{n:n})f(x_{n:n};\theta)\mathrm{d}x_{n:n}. \tag{3}$$

Differentiating (3) with respect to θ on both sides and replacing $X_{n:n}$ for θ, and simplifying one gets

$$\phi(X_{n:n}) = g(X_{n:n}) + \Big(\frac{X_{n:n}}{nc}\Big)\frac{d}{dX_{n:n}}g(X_{n:n}), \tag{4}$$

which establishes necessity of (2). Conversely given (2), let $k(x_{n:n};\theta)$ be the p.d.f. of pdf of n^{th} order statistic such that

$$g(\theta) = \int_a^\theta \Big[g(x_{n:n}) + \Big(\frac{x_{n:n}}{nc}\Big)\frac{d}{dx_{n:n}}g(x_{n:n})\Big] .k(x_{n:n};\theta)\mathrm{d}x_{n:n}, \tag{5}$$

Since $(\frac{c}{x_{n:n}})^n$ is decreasing integrable and differentiable function on the interval (a, b) with $(\frac{a}{c})^n = 0$, the following identity holds

$$g(\theta) \equiv \Big(\frac{c}{\theta^c}\Big)^n \int_a^\theta \Big[\frac{d}{dx_{n:n}}g(x_{n:n})\Big(\frac{x^c_{n:n}}{c}\Big)^n\Big]\mathrm{d}x_{n:n}. \tag{6}$$

Differentiating integrand $g(x_{n:n})\Big(\frac{x^c_{n:n}}{c}\Big)^n$ with respect to $x_{n:n}$ and simplifying after taking $\frac{d}{dx_{n:n}}\Big(\frac{x^c_{n:n}}{c}\Big)^n$ as one factor one gets (6) as

$$g(\theta) \equiv \int_a^\theta \Big[g(x_{n:n}) + \frac{\Big(\frac{x^c_{n:n}}{c}\Big)^n}{\frac{d}{dx_{n:n}}\Big(\frac{x^c_{n:n}}{c}\Big)^n}\frac{d}{dX_{n:n}}g(X_{n:n})\Big] \cdot\Big[\Big(\frac{c}{\theta^c}\Big)^n\frac{d}{dx_{n:n}}\Big(\frac{x^c_{n:n}}{c}\Big)^n\Big]\mathrm{d}x_{n:n} \tag{7}$$

Substituting derivative of $\Big(\frac{x^c_{n:n}}{c}\Big)^n$ in (7) one gets (7) as

$$g(\theta) \equiv \int_a^\theta \phi(x_{n:n})\Big(\frac{nc}{\theta^{nc}}x^{nc-1}_{n:n}\Big)\mathrm{d}x_{n:n}. \tag{8}$$

where $\phi(X_{n:n})$ is as derived in (4). By uniqueness theorem from (5) and (8)

$$k(x_{n:n};\theta) = \Big(\frac{nc}{\theta^{nc}}x^{nc-1}_{n:n}\Big). \tag{9}$$

Since $(\frac{c}{x_{n:n}})^n$ is decreasing integrable and differentiable function on the interval (a, b) with $(\frac{a}{c})^n = 0$ and since $(\frac{c}{x_{n:n}})^n$ is decreasing function for $-\infty < a < b < \infty$ and $(\frac{a}{c})^n = 0$ is satisfy only when range of $x_{n:n}$ is truncated by θ from right and integrating (9) on the interval (a, θ) on both sides, one gets

$$1 = \int_a^\theta k(x_{n:n};\theta)\mathrm{d}x_{n:n}. \tag{10}$$

For $n = 1$, in (10), $[k(x_{n:n};\theta)]_{n=1}$ reduces to $f(x;\theta)$ defined in (1). Hence sufficiency of (2) is established.

Remark 2.1

Using $\phi(X_{n:n})$ derived in (4), the $f(x;\theta)$ given in (1) can be determined by

$$M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}}g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n}))}. \tag{11}$$

and pdf is given by

$$f(x;\theta) = \Big[\frac{\frac{d}{dX_{n:n}}T(X_{n:n})}{T(\theta)}\Big]_{n=1}. \tag{12}$$

where $T(X_{n:n})$ is increasing function for $-\infty < a < b < \infty$ with $T(a) = 0$ such that it satisfies

$$M(X_{n:n}) = \frac{d}{dX_{n:n}}\log\Big[T(X_{n:n})\Big]. \tag{13}$$

Remark 2.2

The theorem 2.1 for function of n^{th} order statistics with remark 2.1 also holds for random variable X when $n = 1$ (see Bhatt [17]).

Page NO. 49

3 Illustrative Examples

Examples

Using method describe in remark 2.1 power function distribution through expectation of non-constant function of order statistics is characterized for illustrative example and significant of unified approach of characterization result.

Example 3.1 Characterization of power function distribution through the Minimum Variance Unbiased (UMVU) estimator $\widehat{\theta^r}$ of θ^r is given.

$$\widehat{\theta^r} = \left(1 + \frac{r}{cn}\right) X_{n:n}^r = g(X_{n:n})$$

Using (4) one gets

$$\phi(X_{n:n}) = \left(1 + \frac{r}{nc}\right)^2 X_{n:n}^r$$

and (11) of remark 2.1 will be

$$M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}} g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n}))} = \frac{nc}{X_{n:n}}.$$

with

$$\frac{d}{dX_{n:n}} \log\left(\frac{X_{n:n}^c}{c}\right)^n = M(X_{n:n}),$$

then

$$T(X_{n:n}) = \left(\frac{X_{n:n}^c}{c}\right)^n,$$

and

$$f(x;\theta) = \left[\frac{\frac{d}{dx_{n:n}} T(x_{n:n})}{T(\theta)}\right]_{n=1} = c\theta^{-c} x^{c-1}.$$

Example 3.2 Characterization of power function distribution through the uniformly minimum variance unbiased (UMVU) estimator $\widehat{g}(\theta)$ and maximum likelihood estimator (MLE) $\widetilde{g}(\theta)$ of $g(\theta)$ such as mean; $\mu_1'(\theta)$, rth moment; $\mu_r'(\theta)$, e^{θ}, $e^{-\theta}$, p^{th} quantile; $Q_p(\theta)$, distribution function; $F(t)$; reliability function; $\bar{F}(t)$, hazard rate; $\lambda(t)$ is given. The UMVU estimator

$$\widehat{g}(\theta) = \begin{cases} \widehat{\mu_1'}(\theta) = \left(c + \frac{1}{n}\right) \frac{X_{n:n}}{c+1}; & \text{for i} = 1, \\ \widehat{\mu_r'}(\theta) = \left(c + \frac{r}{n}\right) \frac{X_{n:n}^r}{c+r}; & \text{for i} = 2, \\ \widehat{e^{\theta}} = \left[1 + \frac{X_{n:n}}{nc}\right] e^{X_{n:n}}; & \text{for i} = 3, \\ \widehat{e^{-\theta}} = \left[1 - \frac{X_{n:n}}{nc}\right] e^{-X_{n:n}}; & \text{for i} = 4, \\ \widehat{Q_p}(\theta) = \left(1 + \frac{1}{nc}\right) p^{-\frac{1}{c}}; & \text{for i} = 5, \\ \widehat{F}(t) = \left(1 - \frac{1}{n}\right)\left(\frac{t}{X_{n:n}}\right)^c; & \text{for i} = 6, \\ \widehat{\bar{F}}(t) = 1 - \left(1 - \frac{1}{n}\right)\left(\frac{t}{X_{n:n}}\right)^c; & \text{for i} = 7, \\ \widehat{\lambda}(t) = -\left[n\left(\left(\frac{t}{X_{n:n}}\right)^c - 1\right)\right]^{-2} \\ \quad \cdot\left[n - 1 + n\left(\frac{t}{X_{n:n}}\right)^c\right] \\ \quad \cdot\left(\frac{c}{t}\right)\left(\frac{t}{X_{n:n}}\right)^c; \text{for i} = 7, \end{cases}$$

and MLE

$$\widetilde{g}(\theta) = \begin{cases} \widetilde{\mu_1'}(\theta) = \frac{c}{c+1} X_{n:n}; \\ \qquad \text{for i} = 9, \\ \widetilde{\mu_r'}(\theta) = \frac{c}{c+r} X_{n:n}^r; \\ \qquad \text{for i} = 10, \\ \widetilde{e^{\theta}} = e^{X_{n:n}}; \\ \qquad \text{for i} = 10, \\ \widetilde{e^{-\theta}} = e^{-X_{n:n}} \\ \qquad \text{for i} = 10, \\ \widetilde{Q_p}(\theta) = p^{-\frac{1}{c}} X_{n:n}; \\ \qquad \text{for i} = 13, \\ \widetilde{F}(t) = \left(\frac{t}{X_{n:n}}\right)^c; \\ \qquad \text{for i} = 14, \\ \widetilde{\bar{F}}(t) = 1 - \left(\frac{t}{X_{n:n}}\right)^c; \\ \qquad \text{for i} = 15, \\ \widetilde{\lambda}(t) = \frac{\frac{c}{t}\left(\frac{t}{X_{n:n}}\right)^c}{1 - \left(\frac{t}{X_{n:n}}\right)^c}; \\ \qquad \text{for i} = 16, \end{cases}$$

one gets

$$[\phi_i(X_{n:n})-\widehat{g}_i(\theta)] = \begin{cases} \left(c+\frac{r}{n}\right)\frac{rX^r_{n:n}}{nc(c+r)}; \text{for } i=1, \\ \left(c+\frac{1}{n}\right)\frac{X_{n:n}}{nc(c+1)}; \text{for } i=2, \\ \left[1+\frac{1}{nc}+\frac{X_{n:n}}{nc}\right] \frac{X_{n:n}}{nc}e^{X_{n:n}}; \text{for } i=3, \\ \left[\frac{X_{n:n}}{nc}-\frac{1}{nc}-1\right] \frac{X_{n:n}}{nc}e^{X_{n:n}}; \text{for } i=4, \\ \frac{X_{n:n}p^{-\frac{1}{c}}}{nc}\left[1+\frac{1}{nc}\right]; \text{ for } i=5, \\ -\frac{1}{n}\left[1-\frac{1}{n}\right]\left(\frac{t}{X_{n:n}}\right)^c; \text{for } i=6, \\ \frac{1}{n}\left[1-\frac{1}{n}\right]\left(\frac{t}{X_{n:n}}\right)^c; \text{for } i=7, \\ n^{-2}\left[\left(\frac{t}{X_{n:n}}\right)^c-1\right]^{-3} \cdot\left[n-1-(n+1)\left(\frac{t}{X_{n:n}}\right)^c\right] \cdot\left(\frac{c}{t}\right)\left(\frac{t}{X_{n:n}}\right)^c; \text{ for } i=8, \end{cases}$$

and

$$[\phi_i(X_{n:n})-\overline{g}_i(\theta)] = \begin{cases} \frac{X_{n:n}}{n(c+1)} \text{ ; for } i=9, \\ \frac{rX^r_{n:n}}{n(c+r)} \text{ ; for } i=10, \\ \frac{X_{n:n}}{nc}e^{-X_{n:n}} \text{ ; for } i=11, \\ -\frac{X_{n:n}}{nc}e^{-X_{n:n}} \text{ ; for } i=12, \\ \frac{X_{n:n}P^{-\frac{1}{c}}}{nc} \text{ ; for } i=13, \\ -\frac{1}{n}\left(\frac{t}{X_{n:n}}\right)^c; \text{ for } i=14, \\ \frac{1}{n}\left(\frac{t}{X_{n:n}}\right)^c; \text{ for } i=15, \\ -\frac{\frac{c}{t}\left(\frac{t}{X_{n:n}}\right)^c}{n\left[1-\left(\frac{t}{X_{n:n}}\right)^c\right]^2} \text{ ; for } i=16, \end{cases}$$

respectively.

Then by defining $M(X_{n:n})$ given in (11) and substituting $T(X_{n:n})$ as appeared in (13) for (12), $f(x;\theta)$ is characterized.

Example 3.3 In context of remark 2.2 characterization of power function distribution through hazard rate; $\lambda(\theta)$ is given.

$$\lambda(\theta) = \frac{\frac{c}{t}\left(\frac{t}{\theta}\right)^c}{1-\left(\frac{t}{X}\right)^c}$$

Therefore

$$g(X) = \frac{\frac{c}{t}\left(\frac{t}{X}\right)^c}{1-\left(\frac{t}{X_{n:n}}\right)^c}$$

and from (4)

$$\phi(X) = g(X) + \left(\frac{X}{c}\right)\frac{d}{dX}g(X) = \frac{\left(\frac{c}{t}\right)\left(\frac{t}{X}\right)^{2c}}{\left[\left(\frac{t}{X}\right)-1\right]^2}$$

and (11) of remark 2.1 will be

$$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X)-g(X)} = \frac{c}{x}$$

with

$$\frac{d}{dX}\log\left[\frac{X^c}{c}\right] = M(X)$$

then

$$T(X) = \frac{X^c}{c}.$$

and

$$f(x;\theta) = \left[\frac{\frac{d}{dx}T(x)}{T(\theta)}\right] = c\theta^{-c}x^{c-1}$$

Example 3.4 Using remark 2.2 the pdf $f(x;\theta)$ defined in (1) can be characterized through non constant functions of θ such as

$$g_i(\theta) = \begin{cases} \frac{c}{c+1}\theta; mean \text{for } i=9, \\ \frac{c}{c+r}\theta^r_{n:n}; r^{th} rowmoment \text{for } i=10, \\ e^{\theta}; \text{for } i=10, \\ e^{-\theta} \text{ for } i=10, \\ p^{-\frac{1}{c}}\theta; p^{th} quantile \text{for } i=13, \\ \left(\frac{t}{\theta}\right)^c; distribution function \text{for } i=14, \\ 1-\left(\frac{t}{\theta}\right)^c; reliability \text{for } i=15, \\ \frac{\frac{c}{t}\left(\frac{t}{\theta}\right)^c}{1-\left(\frac{t}{X}\right)^c}; \text{for } i=16, \end{cases}$$

Page NO. 51

by using

$$[\phi_i(X) - g_i(X] = \begin{cases} \frac{X}{c+1} \text{ ; for i} = 9, \\ \frac{cX^r}{c+r} \text{ ; for i} = 10, \\ \frac{X}{c}e^X \text{ ; for i} = 11, \\ -\frac{X}{c}e^{-X} \text{ ; for i} = 12, \\ \frac{XP^{-\frac{1}{c}}}{c} \text{ ; for i} = 13, \\ -\left(\frac{t}{X}\right)^c \text{; for i} = 14, \\ \left(\frac{t}{X}\right)^c \text{; for i} = 15, \\ \frac{\frac{c}{t}\left(\frac{t}{X}\right)^c}{\left[1-\left(\frac{t}{X}\right)^c\right]^2} \text{ ; for i} = 16, \end{cases}$$

defining $M(X)$ given in (11) and substituting $T(X)$ as appeared in (13) for (12), $f(x;\theta)$ is characterized.

Acknowledgements

This work was supported by UGC Major Research Project : F.No.42-39/2013(SR) dated 12-3-2013.

REFERENCES

[1] A. P. Basu. On characterizing the exponential distribution by order statistics, Ann. Inst. Statist. Math.,17, 93-96, 1965.

[2] Z. Govindarajulu. Characterization of exponential and power distribution, Skand, Aktuarietidskr., 49, 132-136,1966.

[3]]M. M. Desu. A characterization of the exponential distribution by order statistics.Ann. Math. Stat., 42, 837 - 838. 1971.

[4]] A. C. Dallas. Characterization Pareto and power distribution, Ann. Ins. Statist. Math, 28, Part A, pp. 491 - 497, 1976 .

[5] M. L. Beg and Kirmani, S. N. U. A. on a characterization of exponential and related distributions, Austral. J. Statist., 16 (3), 163-166, 1974.

[6] H. N. Nagaraja. On a characterization based on record valves Austral. J. Statist., 19 (1), 70-73, 1977.

[7] M. A. Ali, and A. H. Khan. Characterization of Some Types of Distributions, Information and Management Sciences, Vol. 9, No. 2, June, 1998.

[8] M. Tavangar and M. Asadi. some new characterization results on exponential and related distributions, Bulletin of the Iranian Mathematical Society Vol. 36 No. 1, 257-272, 2010.

[9] M. Faizan and M. I. Khan. A Characterization of Continuous Distributions through Lower Record Statistics, ProbStat Forum, Volume 04, Pages 39-43, April 2011.

[10] M. Fisz. Characterization of some probability distribution,Skand. Aktuarietidskr, 41, 65-67, 1958.

[11] G. Arslan. Characterization based on product of order statistics, math.ST., arXiv 1110.2879v 1, 2011.

[12] M. H. Alamatsaz. Separation of charged and neutral current events from background, proceeding of annual physics conference of Iran, Isfahan University of Technology,1985.

[13] S. Kotz and F. W. Steutel. Note on a characterization of exponential distributions. Stat. Probab. Lett. 6, 201-203, 1988.

[14] A. A. Alzaid and M. A. Al-Osh. An Integer-Valued pth-Order Autoregressive Structure (INAR(p)) Process. J. Appl. Prob., 27: 314-324, 1990.

[15] T. S. K. Moothathu. Characterization of power function distribution through property of Lorenz curve, Sankhya ?, Indian Journal of Statistics, Volume 48, Series B, Pt. 2, 262-265, 1986.

[16] M. G. Kendall and A. Stuart. The Advanced Theory of Statistics,London, Griffin, 1952.

[17] M. B. bhatt. Characterization of Power-Function Distribution through Expectation, Open Journal of Statistics, 3, 441-443, 2013.

Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. **2(2)**, 16-19, February **(2014)** *Res. J. Mathematical and Statistical Sci.*

Characterization of Uniform Distribution u (0,θ) through Expectation

Bhatt Milind. B
Department of Statistics, Sardar Patel University, Vallabhvidyanagar-388129, Dist. Anand, Gujarat, INDIA

Available online at: www.isca.in
Received 5th January 2014, revised 30th January 2014, accepted 11th February 2014

Abstract

For characterization of uniform distribution one needs any arbitrary non constant function only in place of approaches such as independence of sample mean and variance, correlation of minimum and maximum in a random sample of size two, moment conditions, inequality of Chernoff, available in the literature. Path breaking different approach based on identity of distribution and equality of expectation of function of random variable was used in characterizing uniform distribution through expectation of non constant function of random variable with examples for illustrative purpose.

Keywords: Characterization, distribution, correlation, expectation, variable.

Introduction

Characterizations was independently develop in different branches of applied probability and pure mathematics. Characterizations theorem are located on borderline between probability theory and mathematical statistics. It is of general interest to mathematical community, to probabilists and statistician as well as to researchers and practitioner industrial engineering and operation research and various scientist specializing in natural and behavior science, in particular those who are interested in foundation and application of probabilistic model building[1,2].

Geary[3] stated that given sample of size $n \geq 2$ independent observations come from some distribution on the line then sample mean and variance are independent is the necessary and sufficient condition for to be from normally distribution. The need of some regularity condition for Geary's characterization of normal distribution have been removed by successive refinement[2]. Similar characterization for uniform distribution by Kent[4] asserted that if $n \geq 2$ i.i.d random angels from distribution defined by density on circle, sample mean direction and resultant length are independent if and only if angels come from uniform distribution.

Various approaches for characterization of uniform distribution are available in the literature. It is well known that minimum and maximum in a random sample of size two are positively correlate and coefficient of correlation is less or equal to one half. Bartoszyn'ski[5] proposed that a result of this type might exist in connection with a problem in cell division. Since the two daughter cells cannot always be distinguished later, the times till their further division can only be recorded as the earlier event and the later event. The correlation between these ordered pairs thus may provide the only information on the independence of the two events. Terreel[6] showed that the coefficient of correlation is one half if and only if random sample comes from rectangular distribution. Terreel's proof is computational nature and use properties of Legendre polynomial. Lopez-Bldzquez[7] gave ease proof for Terreel's characterization and obtained shaper bound on the coefficient of correlation.

Uniform distribution $U(0,1)$ is neatly characterized by two moment conditions: $E[Max(X_1, X_2)] = \frac{2}{3}$ and $E(X_1^2) = \frac{1}{3}$ by Lin[8]. Using two suitable moments of order statistics Too (1989) characterize uniform and exponential distribution[9-13]. Huang[16] studied density estimation by wavelet-based reproducing kernels and further doing error analysis for bias reduction in a spline-ased multi resolution, Chow[17] (1999) studied *n*-fold convolution modulo one and characterize uniform distribution on interval zero to one.

Inequality of Chernoff[18,19]. assert that" if X is normally distributed with mean 0 and variance 1 and if g is absolutely continuous and g(X) has finite variance, then" $E[\{g'(X)\}^2] \geq V[g(X)]$ and equality holds if and only $g(X)$ is linear. Chernoff proved this result using Hermite polynomials where as Chris[20] proved inequality of Chernoff by using Cauchy- Schwarz inequality and Fubini's theorem. Sumrita[21] (1990) studied Chernoff-type inequalities for distributions on [-1,1] having symmetric unimodal densities and gave characterization of uniform distributions by inequalities of chernoff-type.

This research note provides characterization of uniform distribution with probability density function (pdf)

$$f(x,\theta) = \begin{cases} \frac{1}{\theta}\,; & a < x < \theta < b; \\ 0\,; & otherwise. \end{cases} \qquad (1.1)$$

Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. 2(2), 16-19, February (2014) *Res. J. Mathematical and Statistical Sci.*

where $-\infty \le a < b \le \infty$ are known constants and $(1/\theta)$ is everywhere differentiable function. Since range is truncated by θ from right $a = 0$.

The characterization of uniform distribution through expectation of function, $\phi(X)$ in section 2 and section 3 is for illustrative examples.

Characterization Theorem: Let X be a random variable with distribution function F. Assume that F is continuous on the interval $[a, b]$, where $-\infty \le a < b \le \infty$. Let $\phi(X)$ and $g(X)$ be two distinct differentiable and intregrable functions of X on the interval $[a, b]$ where $-\infty \le a < b \le \infty$. Then

$$E\left[g(X) + X\frac{d}{dx}g(X)\right] = g(\theta). \qquad (2.1)$$

is the necessary and sufficient condition for pdf $f(x,\theta)$ of F to be $f(x,\theta)$ defined in (1.1).

Proof. Given $f(x,\theta)$ defined in (1.1), for necessity of (2.1) if $\phi(X)$ is such that $g(\theta) = E[\phi(X)]$ where $g(\theta)$ is differentiable function then

$$g(\theta) = \int_a^\theta \phi(x) f(x,\theta)\, dx \qquad (2.2)$$

Differentiating with respect to θ on both sides of (2.1) and replacing X for θ and after simplification

$$\phi(X) = g(X) + X\frac{d}{dx}g(X) \qquad (2.3)$$

which establishes necessity of (2.1). Conversely given (2.1), let $k(x,\theta)$ be pdf of r.v. X such that

$$g(\theta) = \int_a^\theta \left[g(x) + x\frac{d}{dx}g(x)\right] k(x,\theta)\, dx. \qquad (2.4)$$

Since $(1/\theta)$ is decreasing on the interval (a,θ) where $-\infty \le a < b \le \infty$ following identity holds :

$$g(\theta) = \frac{1}{\theta}\int_a^\theta \left[\frac{d}{dx}xg(x)\right] dx. \qquad (2.5)$$

Differentiating integrand of (2.5) one gets

$$g(\theta) = \int_a^\theta \left[g(x) + x\frac{d}{dx}g(x)\right]\frac{1}{\theta} dx. \qquad (2.6)$$

And (2.6) will be

$$g(\theta) = \int_a^\theta \phi(x)\frac{1}{\theta} dx \qquad (2.7)$$

where $\phi(x)$ is function of X derived in (2.3). From (2.4) and (2.7) by uniqueness theorem

$$k(x,\theta) = \frac{1}{\theta}. \qquad (2.8)$$

Since $(1/\theta)$ is decreasing on the interval (a,θ) where $-\infty \le a < b \le \infty$ and since $a = 0$ integrating (2.8) on both sides one gets

$$1 = \int_a^\theta k(x,\theta) dx. \qquad (2.9)$$

Hence $k(x,\theta)$ derived in (2.8) reduces to $f(x,\theta)$ defined in (1.1), which establishes sufficiency of (2.1).

Remark 2.1. Using $\phi(X)$ derived in (2.3), the $f(x,\theta)$ given in (1.1) can be determined by

$$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X) - g(X)} \qquad (2.10)$$

and pdf is given by

$$f(x,\theta) = \frac{\frac{d}{dx}(T(x))}{T(\theta)} \qquad (2.11)$$

where $T(X)$ is increasing function for $-\infty \le a < b \le \infty$ with $T(a) = 0$ such that it satisfies

$$M(X) = \frac{d}{dx}[\log(T(X))]. \qquad (2.12)$$

Applications

Example-1: Using method described in the remark characterization of uniform distribution through p^{th} quantile $Q_p(\theta) = \theta p$ is illustrated.

$g(\theta) = \theta p$
$g(X) = Xp$
$\phi(X) = g(X) + X\frac{d}{dx}g(X) = 2Xp$
$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X) - g(X)} = \frac{1}{X}$
$\frac{d}{dx}[\log(X)] = \frac{1}{X} = M(X)$
$T(X) = X$
$f(x,\theta) = \frac{\frac{d}{dx}(T(x))}{T(\theta)} = \frac{1}{\theta}$

Example-2: Using method described in the remark characterization of uniform distribution through p^{th} quantile $e^{-\theta}$ is illustrated.

$g(\theta) = e^{-\theta}$
$g(X) = e^{-X}$
$\phi(X) = g(X) + X\frac{d}{dx}g(X) = (1 - X)e^{-X}$
$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X) - g(X)} = \frac{1}{X}$
$\frac{d}{dx}[\log(X)] = \frac{1}{X} = M(X)$
$T(X) = X$
$f(x,\theta) = \frac{\frac{d}{dx}(T(x))}{T(\theta)} = \frac{1}{\theta}$

Example-3: The pdf $f(x,\theta)$ defined in (1.1) can be characterized through non constant function such as

Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. **2(2)**, 16-19, February **(2014)** *Res. J. Mathematical and Statistical Sci.*

$$g_i(\theta) = \begin{cases} \frac{\theta}{2}; for\ i = 1, Mean, \\ \frac{\theta^r}{r+1}; for\ i = 2,\ \text{r}^{\text{th}}\ \text{raw moment}, \\ e^{\theta}\ ; for\ i = 3, \\ e^{-\theta}; for\ i = 4, \\ \theta p; for\ i = 5,\ \text{p}^{\text{th}}\ \text{quantile}, \\ \frac{t}{\theta}; for\ i = 6,\ \text{distribution function at t}, \\ 1 - \frac{t}{\theta}; for\ i = 7,\ \text{Relibility at t}, \\ 1 - \frac{t}{\theta}\ ; for\ i = 8,\ \text{Hazard function}, \end{cases}$$

and by using

$$[\phi_i(X) - g_i(X)] = \begin{cases} \frac{x}{2}; for\ i = 1, Mean, \\ \frac{r}{r+1}x^r\ ; for\ i = 2,\ \text{r}^{\text{th}}\ \text{raw moment}, \\ xe^{x}; for\ i = 3, \\ -xe^{-x}; for\ i = 4, \\ xp; for\ i = 5,\ \text{pth quantile}, \\ -\frac{t}{x}; for\ i = 6,\ \text{distribution Function at t}, \\ \frac{t}{x}; for\ i = 7,\ \text{Relibility Function at t}, \\ \frac{x}{(x-t)^2}; for\ i = 8,\ \text{Hazard Function}, \end{cases}$$

and defining $M(X)$ given in (2.10) and substituting $T(X)$ as appeared in (2.12) for (2.11).

Note that to characterize pdf given in (1.1) one needs any arbitrary non constant function only.

Conclusion

To characterize pdf given in (1.1) one needs any arbitrary non constant function only.

Reference

1. Lukacs E. and Laha R.G., Application of characteristic function, Griffin, London, **(1964)**
2. Kagan A.M., Linnik, Yu.V. and Rao C.R., Characterization Problems in Mathematical Statistics, J. Wiley, New York (English translation), **(1973)**
3. Geary R.C., The distribution of sStudent ratio for Non-Normal Samples, *J. Roy. Statis. Soc. Ser. B* **3**, 178-184 **(1936)**
4. Kent J.T., Mardia K.V. and Rao. J.S., Characterization of the Uniform Distribution on the Circle, *The Annals of Statistics,* **7(4),** 882-889 **(1979)**
5. Bartoszynski R., Personal Communication, **(1980)**
6. Terrel G.R., A characterization of Rectangular Distributions, *The Annals of Probability,* **11(3),** 823-826 **(1983)**
7. Lopez –Bldzquez F. and B. Salamanca Milño, On Terrers Characterization of Uniform Distribution, *Statistical Papers,* **40**, 335-342 **(1999)**
8. Lin G. D, Characterizations of Uniform Distributions and of Exponential Distributions, Sankhy¯a A, **50,** 64–69 **(1988)**
9. Stapleton J.H., A Characterization of the Uniform Distribution on a Compact Topological Group, *Ann. Math. Statist.,* **34**, 319-326 **(1963)**
10. Arnold B.C. and Meeden G., A Characterization of the Uniform Distribution Based on Summation Modulo one, with Application to Fractional Backlogs, *Austral. J. Statist.,* **18(3)**, 173-175 **(1976)**
11. Driscoll M.F, On Pair-wise and Mutual Independence: Characterizations of Rectangular Distributions, *J. Amer. Statist. Assoc.,* **73**, 432-433 **(1978)**
12. Shimizu R. and Huang J.S., On a Characteristic Property of the Uniform Distribution, *Ann. Inst. Statist. Math.,* **35**, 91-94 **(1983)**
13. Abdelhamid S.N., On a Characterization of Rectangular Distributions, *Statistics and Probability Letters,* **3**, 235-238 **(1985)**
14. Rogers G.S, Alternative Proof of the Characterization of the Density AX^B, *The American Mathematical Monthly,* **70(8),** 857-858 **(1963)**
15. Srivastava M.S., A Characterization of the Exponential Distribution, *The American Mathematical Monthly,* **74(4)**, 414-416 **(1967)**
16. Huang S.Y., Density Estimation by Wavelet-based Reproducing Kernels, *Statistica Sinica,* **9**, 137-151 **(1999)**
17. Chow Y. and Huang S.Y., A Characterization of the Uniform Distribution via Moments of n-fold Convolution

Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. **2(2)**, 16-19, February (**2014**) *Res. J. Mathematical and Statistical Sci.*

Modulo one, Sankhy¯a : *The Indian Journal of Statistics*, **61**, Series A, Pt., **1**, 148-151(**1999**)

18. Chernoff H., The Identification of an Element of a Large Population in the Presence of Noise, *Ann. Statist*, **8** 1179-1197 (**1980**)

19. Chernoff H., A Note on an Inequality Involving the Normal Distribution, *Ann. Probab.*, **9**, 533-535 (**1981**)

20. Chris A.J. Klaassen, On an Inequality of Chernoff, *The Annals of Probability*, **13(3)**, 966-974 (**1985**)

21. Sumitra P. and Subir, B., Characterization of uniform distributions by Inequalities of Chernoff-type. Sankhy$\bar{a}$: *The Indian Journal of Statistics*, **52**, Series A, Pt. **3**, 376-382 (**1990**)

Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. 2(8), 10-14, August (2014) Res. J. Mathematical and Statistical Sci.

Characterization of Uniform Distribution through Expectation of Function of Order Statistics

Bhatt Milind. B
Department of Statistics, Sardar Patel University, Vallabhvidyanagar-388129, Dist. Anand, Gujarat, INDIA

Available online at: www.isca.in, www.isca.me
Received 12th July 2014, revised 1st August 2014, accepted 12th August 2014

Abstract

For characterization of uniform distribution one needs any arbitrary non constant function only in place of approaches such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy etc. available in the literature. Recently Bhatt characterized negative exponential distribution through expectation of non constant function of random variable. Attempt is made to extend the characterization of negative exponential distribution through expectation of any arbitrary non constant function of order statistics.

Keyword: Characterization; Uniform distribution. **MSC 2010 Subject Clasification :** 62E10

Introduction

Characterizations theorem are located on borderline between probability theory and mathematical statistics. It is of general interest to mathematical community, to probabilists and statistician as well as to researchers and practitioner industrial engineering and operation research and various scientist specializing in natural and behavior science, in particular those who are interested in foundation and application of probabilistic model building. (see basic book on characterizations by Lukacs and Laha[1] and the more advance comprehensive mathematical tools (entirely toward normal distribution) see kagan, Linnik and Rao[2]).

Various approaches for characterization of uniform distribution are available in the literature. It is well known that smaller and the larger of a random sample of size two are positively correlate and coefficient of correlation is less or equal to one half. Bartoszyn'ski[3] proposed that a result of this type might exist in connection with a problem in cell division. Since the two daughter cells cannot always be

This work is supported by UGC Major Research Project No: F.No.42-39/2013 (SR), dated 12-3-2013. Distinguished later, the times till their further division can only be recorded as the earlier event and the later event. The correlation between these ordered pairs thus may provide the only information on the independence of the two events. Terreel[4] showed that the coefficient of correlation is one half if and only if random sample comes from rectangular distribution. Terreel's proof is computational nature and use properties of Legendre polynomial. Lopez –Bldzquez[5] gave ease proof for Terreel's characterization and obtained shaper bound on the coefficient of correlation.

Geary[6] stated that given sample of size $n \geq 2$ independent observations come from some distribution on the line then sample mean and variance are independent if and only if observations are normally distributed. The need of some regularity condition for Geary's characterization of normal distribution have been removed by successive refinement (see kagan, Linnik and Rao[2] page. 103). Similar characterization for uniform distribution by Kent[7] asserted that if $n \geq 2$ i.i.d random angels from distribution defined by density on circle, sample mean direction and resultant length are independent if and only if angels come from uniform distribution.

Uniform distribution $U(0,1)$ is neatly characterized by two moment conditions: $E[Max(X_1, X_2)] = 2/3$ and $E(X_1^2) = 1/3$ by Lin[8]. Using two suitable moments of order statistics Too[9] characterize uniform and exponential distribution. Other contributions concerning use of property of maximal correlation coefficient between order statistics, of identically distributed spacings etc [see Stapleton[10], Arnold[11], Driscoll[12], Shimizu[13], Abdelhamid[14]].

Huang[15] studied density estimation by wavelet-based reproducing kernels and further doing error analysis for bias reduction in a spline-ased multi resolution, Chow[16] characterized uniform distribution $U(0,1)$ via moments of n-fold convolution modulo one.

Inequality of Chernoff[17,18] assert that if X is normally distributed with mean 0 and variance 1 and if g is absolutely continuous and g(X) has finite variance, then $E[\{g'(X)\}^2] \geq V[g(X)]$ and equality holds if and only $g(X)$ is linear. Chernoff proved this result using Hermite polynomials where as Chris[19] proved inequality of Chernoff by using Cauchy- Schwarz inequality and Fubini's theorem. Sumrita[20] studied Chernoff-type inequalities for distributions on [-1,1] having symmetric

Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. 2(8), 10-14, August (2014) *Res. J. Mathematical and Statistical Sci.*

unimodal densities and gave characterization of uniform distributions by inequalities of chernoff-type.

Using identity and equality of expectation of function of order statistics, this research note provides path breaking new characterization of uniform distribution with probability density function (pdf)

$$f(x,\theta) = \begin{cases} \frac{1}{\theta}\ ;\ a < x < \theta < b; \\ 0\ ;\ otherwise. \end{cases} \qquad \ldots(1.1)$$

where $-\infty \le a < b \le \infty$ are known constants and $\left(\frac{1}{\theta}\right)$ is everywhere differentiable function. Since range is truncated by θ from right $a = 0$.

The aim of the present research note is to give a new characterization through expectation of function of order statistics, $\phi(.)$ for uniform distribution. The characterization theorem given in section 2 and section 3 is devoted to applications for illustrative purpose.

Characterization

Theorem: Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from distribution function F. Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ be the corresponding set of order statistics. Assume that F is continuous on the interval (a,b), where $-\infty \le a < b \le \infty$. Let $\phi(X_{n:n})$ and $g(X_{n:n})$ be two distinct differentiable and intregrable functions of n^{th} order statistic; $X_{n:n}$, on the interval (a,b), where $-\infty \le a < b \le \infty$ and moreover $g(X_{n:n})$ be non-constant of $X_{n:n}$. Then

$$\mathrm{E}\left[g(X_{n:n}) + \left(\frac{X_{n:n}}{n}\right)\frac{d}{dX_{n:n}}g(X_{n:n})\right] = \mathrm{g}(\theta) \qquad \ldots(2)$$

is the necessary and sufficient condition for pdf $f(x,\theta)$ of F to be $f(x,\theta)$ defined in (1).

Proof : Given $f(x,\theta)$ defined in (1), for necessity of (2) if $\phi(X_{n:n})$ is such that $g(\theta) = E[\phi(X_{n:n})]$ where $g(\theta)$ is differentiable function then using $f(x_{n:n},\theta)$; pdf of nth order statistics one gets

$$\mathrm{g}(\theta) = \int_a^\theta \phi(x_{n:n})\, f(x_{n:n},\theta)\, dx_{n:n} \qquad (3)$$

Differentiating with respect to θ on both sides of (2.2), replacing $X_{n:n}$ for θ and simplifying the result one gets

$$\phi(X_{n:n}) = g(X_{n:n}) + \left(\frac{X_{n:n}}{n}\right)\frac{d}{dX_{n:n}}g(X_{n:n}) \qquad (4)$$

which establishes necessity of (2). Conversely given (2), let $k(x_{n:n},\theta)$ be such that

$$g(\theta) = \int_a^\theta \left[g(X_{n:n}) + \left(\frac{X_{n:n}}{n}\right)\frac{d}{dX_{n:n}}g(X_{n:n})\right] k(x_{n:n},\theta)\, dx_{n:n}. \qquad (5)$$

Since $(1/x_{n:n})^n$ is decreasing integrable and differentiable function on the interval (a,b) with $a^n = 0$ the following identity holds.

$$g(\theta) \equiv \left(\frac{1}{\theta}\right)^n \int_a^\theta \left[\frac{d}{dx_{n:n}}\{x_{n:n}^n g(x_{n:n})\}\right] dx_{n:n} \qquad (6)$$

Differentiating $\{x_{n:n}^n g(x_{n:n})\}$ with respect to $x_{n:n}$ and simplifying (6) after taking $\left\{\frac{d}{dx_{n:n}} x_{n:n}^n\right\}$ as one factor, one gets (6) as

$$g(\theta) \equiv \int_a^\theta \left[g(x_{n:n}) + \frac{x_{n:n}^n}{\frac{d}{dx_{n:n}}(x_{n:n}^n)}\frac{d}{dx_{n:n}}g(x_{n:n})\right]\left\{\left(\frac{1}{\theta}\right)^n \frac{d}{dx_{n:n}}\{x_{n:n}^n\}\right\} dx_{n:n} \qquad (7)$$

and substituting derivative of $\frac{d}{dx_{n:n}} x_{n:n}^n$ in (2.6) one gets (7) as

$$g(\theta) \equiv \int_\theta^b \phi(x_{n:n}) \left\{n\left(\frac{1}{\theta}\right)^n x_{n:n}^{n-1}\right\} dx_{n:n} \qquad (8)$$

where $\phi(X_{n:n})$ is derived in (4).

By uniqueness theorem from (5) and (8)

$$k(x_{n:n},\theta) = n\left(\frac{1}{\theta}\right)^n x_{n:n}^{n-1}. \qquad (9)$$

Since $(1/x_{n:n})^n$ is decreasing increasing integrable and differentiable function on the interval (a,b) where $-\infty \le a < b \le \infty$ and since $\left[\frac{d}{dx_{n:n}} x_{n:n}^n\right]$ is positive intregrable function on the interval (a,b) where $-\infty \le a < b \le \infty$ with $a^n = 0$ and integrating (9) over the interval (a,θ) on both sides, one gets (9) as

$$\mathrm{k}(x_{n:n};\theta) = \left(\frac{1}{\theta}\right)^n \frac{d}{dx_{n:n}} x_{n:n}^n\ ;\ a < x_{n:n} < \theta < b \qquad (10)$$

and

$$1 = \int_a^\theta \mathrm{k}(x_{n:n};\theta)\, dx_{n:n}. \qquad (11)$$

The equation (10), $[\mathrm{k}(x_{n:n};\theta)]_{n=1}$ reduces to f(x; θ) defined in (1) which establishes sufficiency of (2).

Remark 2.1 Denoting

$$M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}}g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n})} \qquad (12)$$

one can determine $f(x,\theta)$ given in (1) as

Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. 2(8), 10-14, August (2014) Res. J. Mathematical and Statistical Sci.

$$f(x,\theta) = \left[\frac{\frac{d}{dx_{n:n}}(T(x_{n:n}))}{T(\theta)}\right]_{n=1} \quad (13)$$

where $T(X_{n:n})$ is decreasing function for $-\infty \le a < b \le \infty$ with $T(a) = 0$ such that it satisfies

$$M(X_{n:n}) = \frac{d}{dX_{n:n}}\left[\log(T(X_{n:n}))\right]. \quad (14)$$

Remark. 2.2 The theorem 2.1 for function of n[th] order statistics also holds for function of random variable X when $(n = 1)$.

Examples

Examples: Consider the uniformly minimum variance unbiased estimator, $\widehat{\theta^r}$ of θ^r,

$$g(X_{n:n}) = \left(\frac{n+r}{n}\right) X^r_{n:n} = \widehat{\theta^r}$$

$$\phi(X_{n:n}) = g(X_{n:n}) + \left(\frac{x^n_{n:n}}{\frac{d}{dX_{n:n}}x^n_{n:n}}\right)\frac{d}{dX_{n:n}} g(X_{n:n}) = \left(1+\frac{r}{n}\right)^2 X^r_{n:n}$$

$$M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}}g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n})} = \frac{n}{X_{n:n}}$$

$$\frac{d}{dX_{n:n}}\left[\log(x^n_{n:n})\right] = \frac{n}{X_{n:n}} = M(X_{n:n})$$

$$T(X_{n:n}) = x^n_{n:n}$$

$$f(x,\theta) = \left\{\frac{\frac{d}{dX_{n:n}}(T(X_{n:n}))}{T(\theta)}\right\}_{n=1} = \frac{1}{\theta}$$

Examples: Using the uniformly minimum variance unbiased (UMVU) estimator $\hat{g}(\theta)$ and maximum likelihood estimator (MLE) $\tilde{g}(\theta)$ of $g(\theta)$ such as mean; θ, r[th] moment; $\theta^r, e^\theta, e^\theta$, p[th] quantile; $Q_p(\theta)$, distribution function; $F(t)$; reliability function; $\bar{F}(t)$, hazard rate; $\lambda(t)$, one gets $[\phi(X_{n:n}) - g(X_{n:n})]$ as given below

$\mu_1'(\theta)$	$\widehat{\mu_1'(\theta)} = \frac{X_{n:n}}{2}\left[1+\frac{1}{n}\right]$	$\frac{X_{n:n}}{2n}\left[1+\frac{1}{n}\right]$
	$\widetilde{\mu_1'(\theta)} = \frac{X_{n:n}}{2}$	$\frac{X_{n:n}}{2n}\left[1+\frac{1}{n}\right]$
$\mu_r'(\theta)$	$\widehat{\mu_r'(\theta)} = \frac{X^r_{n:n}}{r+1}\left[1+\frac{r}{n}\right]$	$\frac{rX^r_{n:n}}{n(r+1)}\left[1+\frac{r}{n}\right]$
	$\widetilde{\mu_r'(\theta)} = \frac{X^r_{n:n}}{r+1}$	$\frac{r}{n(r+1)}X^r_{n:n}$
e^θ	$\widehat{e^\theta} = \left[1+\frac{X_{n:n}}{n}\right]e^{X_{n:n}}$	$\frac{X_{n:n}}{n}e^{X_{n:n}}\left[1+\frac{X_{n:n}}{n}+\frac{1}{n}\right]$
	$\widetilde{e^\theta} = e^{X_{n:n}}$	$\frac{X_{n:n}}{n}e^{X_{n:n}}$
$e^{-\theta}$	$\widehat{e^{-\theta}} = \left[1-\frac{X_{n:n}}{n}\right]e^{-X_{n:n}}$	$\frac{X_{n:n}}{n}e^{-X_{n:n}}\left[\frac{X_{n:n}}{n}-\frac{1}{n}-1\right]$
	$\widetilde{e^{-\theta}} = e^{-X_{n:n}}$	$-\frac{X_{n:n}}{n}e^{X_{n:n}}$
$Q_p(\theta)$	$\widehat{Q_p}(\theta) = \left[1+\frac{1}{n}\right]X_{n:n}p$	$\left[1+\frac{1}{n}\right]\left(\frac{X_{n:n}}{n}\right)p$
	$\tilde{Q}_p(\theta) = X_{n:n}p$	$\left(\frac{X_{n:n}}{n}\right)p$
$F(t)$	$\hat{F}(t) = \left(\frac{n-1}{n}\right)\frac{t}{X_{n:n}}$	$-\left(\frac{n-1}{n^2}\right)\frac{t}{X_{n:n}}$
	$\tilde{F}(\theta) = \frac{t}{X_{n:n}}$	$-\frac{t}{nX_{n:n}}$
$\bar{F}(t)$	$\hat{\bar{F}}(t) = 1-\frac{t}{X_{n:n}}\left(\frac{n-1}{n}\right)$	$\left(\frac{n-1}{n^2}\right)\frac{t}{X_{n:n}}$
	$\tilde{\bar{F}}(t) = 1-\frac{t}{X_{n:n}}$	$\frac{t}{nX_{n:n}}$
$\lambda(t)$	$\hat{\lambda}(t) = \frac{1}{X_{n:n}-t}\left\{1-\frac{X_{n:n}}{n(X_{n:n}-t)}\right\}$	$-\frac{X_{n:n}(t+nt+X_{n:n}-nX_{n:n})}{n^2(t-X_{n:n})^3}$
	$\tilde{\lambda}(t) = \frac{1}{X_{n:n}-t}$	$-\frac{X_{n:n}}{n(X_{n:n}-t)^2}$

substituting $T(X_{n:n})$ as appeared in (14) for (13).

Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. **2(8)**, 10-14, August **(2014)** *Res. J. Mathematical and Statistical Sci.*

Examples: In context of remark
The pdf $f(x,\theta)$ defined in (1) can be characterized through non constant function such as

$$g_i(\theta) = \begin{cases} \frac{\theta}{2}; for\ i = 1, Mean, \\ \frac{\theta^r}{r+1}; for\ i = 2,\ \text{r}^{\text{th}} \text{ raw moment}, \\ e^{\theta}\ ; for\ i = 3, \\ e^{-\theta}; for\ i = 4, \\ \theta p; for\ i = 5,\ \text{p}^{\text{th}} \text{ quantile}, \\ \frac{t}{\theta}; for\ i = 6,\ \text{distribution function at t}, \\ 1 - \frac{t}{\theta}; for\ i = 7,\ \text{Relibility at t}, \\ 1 - \frac{t}{\theta}\ ; for\ i = 8,\ \text{Hazard function}, \end{cases}$$

and by using

$$[\phi_i(X) - g_i(X)] = \begin{cases} \frac{x}{2}; for\ i = 1, Mean, \\ \frac{r}{r+1}x^r\ ; for\ i = 2,\ \text{r}^{\text{th}} \text{ raw moment}, \\ xe^x; for\ i = 3, \\ -xe^{-x}\ ; for\ i = 4, \\ xp; for\ i = 5,\ \text{pth quantile}, \\ -\frac{t}{x}; for\ i = 6,\ \text{distribution Function at t}, \\ \frac{t}{x}; for\ i = 7,\ \text{Relibility Function at t}, \\ -\frac{x}{(x-t)^2}; for\ i = 8,\ \text{Hazard Function}, \end{cases}$$

and defining $M(X)$ given in (12) and substituting $T(X)$ as appeared in (14) for (13).

Conclusion

To characterize pdf given in (12) one needs any arbitrary non constant function only.

Reference

1. Lukacs E and Laha R.G., Application of characteristic function. Griffin, London **(1964)**
2. Kagan A.M., Linnik Yu. V. and Rao C.R., Characterization Problems in Mathematical Statistics, *J. Wiley*, New York (English translation), **(1973)**
3. Bartoszynski, R., Personal communication **(1980)**
4. Terrel G.R., A characterization of rectangular distributions, *The Annals of Probability*, **11(3)**, 823-826 **(1983)**
5. Lopez –Bldzquez, F. and B. Salamanca Milñо, On Terrers characterization of uniform distribution Statistical Papers, **40**, 335-342, **(1999)**
6. Geary R.C., The distribution of student ratio for non-normal samples, *J. Roy. Statis. Soc. Ser. B*, **3**, 178-184 **(1936)**
7. Kent J.T., Mardia, K.V. and Rao. J.S., Characterization of the uniform distribution on the circle, *The Annals of Statistics*, **7(4)**, 882-889 **(1979)**
8. Lin G.D, Characterizations of uniform distributions and of exponential distributions, Sankhy¯a A, **50**, 64–69 **(1988)**
9. Too Y.H. and Lin G.D., Characterizations of uniform and exponential distributions, Statistics and Probability Letters, **7**, 357-359, **(1989)**
10. Stapleton J.H., A characterization of the uniform distribution on a compact topologi cal group, *Ann. Math. Statist.*, **34**, 319-326, **(1963)**
11. Arnold B.C. and Meeden G., A characterization of the uniform distribution based on summation modulo one, with application to fractional backlogs, *Austral. J. Statist.*, **18(3)**, 173-175 **(1976)**
12. Driscoll M.F., On pairwise and mutual independence: characterizations of rectangular distributions, *J. Amer. Statist. Assoc.*, **73**, 432-433 **(1978)**
13. Shimizu R. and Huang J.S., On a characteristic property of the uniform distribution, *Ann. Inst. Statist. Math.*, **35**, 91-94, **(1983)**
14. Abdelhamid S.N., On a characterization of rectangular distributions, *Statistics and Probability Letters*, **3**, 235-238, **(1985)**
15. Huang S.Y., Density estimation by wavelet-based reproducing kernels, *Statistica Sinica*, 137-151 **(1999)**
16. Chow Y. and Huang Su-yun, A characterization of the uniform distribution via moments of n-fold convolution modulo one. Sankhy¯a : The Indian Journal of Statistics, **61,** *Series A, Pt. 1*, 148-151, **(1999)**
17. Chernoff H., The identification of an element of a large population in the presence of noise, *Ann. Statist.*, **8**, 1179 **(1981)**
18. Chernoff H., A note on an inequality involving the normal distribution, *Ann. Probab.* **9**, 533 **(1981)**

Research Journal of Mathematical and Statistical Sciences ISSN 2320–6047
Vol. **2(8)**, 10-14, August **(2014)** *Res. J. Mathematical and Statistical Sci.*

19. Chris A.J. Klaassen, On an inequality of chernoff. *The Annals of Probability,* **13(3),** 966-974, **(1985)**

20. Sumitra P. and Subir B., Characterization of uniform distributions by inequalities of chernoff-type, Sankhy $\bar{a}$: *The Indian Journal of Statistics,* **52, A,** Pt., **3,** 376-382 **(1990)**

Open Journal of Statistics, 2014, 4, 563-569
Published Online September 2014 in SciRes. http://www.scirp.org/journal/ojs
http://dx.doi.org/10.4236/ojs.2014.48052

Scientific Research

Characterization of Generalized Uniform Distribution through Expectation

Milind B. Bhatt

Department of Statistics, Sardar Patel University, Vallabh Vidhyanagar, India
Email: bhattmilind_b@yahoo.com

Received 29 April 2014; revised 25 May 2014; accepted 15 June 2014

Abstract

Normally the mass of a root has a uniform distribution but some have different uniform distributions named Generalized Uniform Distribution (GUD). The characterization result based on expectation of function of random variable has been obtained for generalized uniform distribution. Applications are given for illustrative purpose including a special case of uniform distribution.

Keywords

Characterization, Generalized Uniform Distribution (GUD)

1. Introduction

Normally the mass of a root has a uniform distribution. Plant develops into the reproductive phase of growth; a mat of smaller roots grows near the surface to a depth of approximately 1/6-th of maximum depth achieve (see G. Ooms and K. L. Moore [1]). Dixit [2] studied the problem of efficient estimation of parameters of a uniform distribution in the presence of outliers. He assumed that a set of random variables $X_1, X_2, \cdots, X_n$ represents the masses of roots where out of *n*-random variables some of these roots (say *k*) have different masses; therefore, those masses have different uniform distributions with unknown parameters and these *k* observations are distributed with Generalize Uniform Distribution (GUD) with probability density function (pdf)

$$f(x;\theta)=\begin{cases}\dfrac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha}; & \alpha>-1, a\le x<\theta\le b, a\equiv 0\\ 0; & \text{otherwise.}\end{cases} \tag{1.1}$$

where $-\infty \le a < b \le \infty$ are known constants; X^{α} is positive absolutely continuous function and $(1/\theta)^{\alpha+1}$ is everywhere differentiable function. Since derivative of $X^{\alpha+1}$ being positive, range is truncated by θ from right $a \equiv 0$.

Dixt [3] obtained Maximum Likelihood Estimator (MLE) and the Uniformly Minimum Variance (UMVU)

How to cite this paper: Bhatt, M.B. (2014) Characterization of Generalized Uniform Distribution through Expectation. *Open Journal of Statistics*, **4**, 563-569. http://dx.doi.org/10.4236/ojs.2014.48052

estimator of reliability functional, $P[X>Y]$ in the same setup and showed that the UMVUE is better than MLE when one parameter of GUD is known, where as both parameters of the GUD are unknown, $P[X>Y]$ is estimated by using mixture estimate and is consistent.

In this paper the problem of characterization of GUD with pdf given in (1.1) has been studied and the characterization also holds for uniform distribution on interval (a,θ) when $\alpha=0$. Various approaches were used to characterize uniform distribution; few of them have used coefficient of correlation of smaller and the larger of a random sample of size two; Bartoszyn'ski [4], Terreel [5], Lopez-Bldzquez [6] as Kent [7], have used independence of sample mean and variance; Lin [8], Too [9], Arnold [10], Driscoll [11], Shimizu [12], and Abdelhamid [13] have used moment conditions, *n*-fold convolution modulo one and inequalities of Chernoff-type were also used (see Chow [14] and Sumrita [15]).

In contrast to all above brief research background and application of characterization of member of Pearson family, this research does not provide unified approach to characterized generalized uniform.

The aim of the present research note is to give a path breaking new characterization for generalized uniform distribution through expectation of function of random variable, $\phi(X)$ using identity and equality of expectation of function of random variable. Characterization theorem was derived in Section 2 with method for characterization as remark and Section 3 devoted to applications for illustrative purpose including special case of uniform distribution.

2. Characterization

Theorem 2.1. Let X be a continuous random variable (rv) with distribution function $F(X)$ having pdf $f(x;\theta)$. Assume that $F(X)$ is continuous on the interval (a,b), where $-\infty\le a<b\le\infty$. Let $g(X)$ be a differentiable functions of X on the interval (a,b), where $-\infty\le a<b\le\infty$ and more over $g(X)$ be non-constant. Then $f(x;\theta)$ is the pdf of Generalize Uniform Distribution (GUD) defined in (1.1) if and only if

$$E\left[g(X)+\left(\frac{X}{\alpha+1}\right)\frac{\mathrm{d}}{\mathrm{d}X}g(X)\right]. \tag{2.1}$$

Proof: Given $f(x;\theta)$ defined in (1.1), if $\phi(X)$ is such that $g(\theta)=E[\phi(X)]$ where $g(\theta)$ is differentiable function then

$$g(\theta)=\int_a^\theta \phi(x)f(x;\theta)\,\mathrm{d}x. \tag{2.2}$$

Differentiating with respect to θ on both sides of (2.2) and replacing X for θ after simplification one gets

$$\phi(X)=g(X)+\left(\frac{X}{\alpha+1}\right)\frac{\mathrm{d}}{\mathrm{d}X}g(X). \tag{2.3}$$

which establishes necessity of (2.1). Conversely given (2.1), let $k(x;\theta)$ be the pdf of rv X such that

$$g(\theta)=\int_a^\theta\left[g(X)+\left(\frac{X}{\alpha+1}\right)\frac{\mathrm{d}}{\mathrm{d}X}g(X)\right]k(x;\theta)\,\mathrm{d}x. \tag{2.4}$$

Since $a=0$, the following identity holds

$$g(\theta)=\left(\frac{\alpha+1}{\theta^{\alpha+1}}\right)\int_a^\theta\left[\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(\frac{x^{\alpha+1}}{\alpha+1}\right)g(X)\right\}\right]\mathrm{d}x. \tag{2.5}$$

Differentiating $\left(\frac{x^{\alpha+1}}{\alpha+1}\right)g(x)$ with respect to x and simplifying after tacking $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{\alpha+1}}{\alpha+1}\right)$ as one factor one gets (2.5) as

$$g(\theta)=\int_a^\theta\left[g(x)+\left\{\frac{\frac{x^{\alpha+1}}{\alpha+1}}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{\alpha+1}}{\alpha+1}\right)}\right\}\frac{\mathrm{d}}{\mathrm{d}X}g(X)\right]\left\{\left(\frac{\alpha+1}{\theta^{\alpha+1}}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{\alpha+1}}{\alpha+1}\right)\right\}\mathrm{d}x. \tag{2.6}$$

Substituting derivative of $\left(x^{\alpha+1}/\alpha+1\right)$ in (2.6) it reduces to

$$g(\theta)=\int_{a}^{\theta}\phi(X)\left\{\left(\frac{\alpha+1}{\theta^{\alpha+1}}\right)x^{\alpha+1}\right\}\mathrm{d}x. \tag{2.7}$$

where $\phi(X)$ is derived in (2.3) and by uniqueness theorem from (2.4) and (2.7)

$$k(x;\theta)=\frac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha+1} \tag{2.8}$$

Since $(1/a)^{\alpha}$ is decreasing function for $-\infty\le a<b\le\infty$ and $a^{\alpha+1}=0$ is satisfy only when range of X is truncated by θ from right and integrating (2.8) on the interval (a,θ) on both sides, one gets $k(x;\theta)$ derived in (2.8) as

$$k(x;\theta)=\frac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha};\ \alpha>-1, a\le x<\theta\le b, a\equiv 0. \tag{2.9}$$

and

$$1=\int_{a}^{\theta}k(x;\theta)\mathrm{d}x$$

Hence $k(x;\theta)$ derived in (2.9) reduces to $f(x;\theta)$ defined in (1.1) which establishes sufficiency of (2.1).

Remark 2.1. Using $\phi(X)$ derived in (2.3), the $f(x;\theta)$ given in (1.1) can be determined by

$$M(X)=\frac{\frac{\mathrm{d}}{\mathrm{d}X}g(X)}{\phi(X)-g(X)}. \tag{2.10}$$

and pdf is given by

$$f(x;\theta)=\frac{\frac{\mathrm{d}}{\mathrm{d}X}T(X)}{T(\theta)}. \tag{2.11}$$

where $T(X)$ is increasing function for $-\infty\le a<b\le\infty$ with $T(a)=0$ such that it satisfies

$$M(X)=\frac{\mathrm{d}}{\mathrm{d}X}\left[\log T(X)\right]. \tag{2.12}$$

Remark 2.2. If $\alpha=0$ characterization theorem 2.1 also holds for uniform distribution with pdf

$$f(x;\theta)=\begin{cases}\frac{1}{\theta}; & a\le x<\theta\le b, a\equiv 0\\ 0; & \text{otherwise.}\end{cases} \tag{2.13}$$

3. Examples

Using method describe in remark 2.1 Generalize Uniform Distribution (GUD) through expectation of non-constant function of random variable such as mean, r^{th} raw moment, e^{θ}, $e^{-\theta}$, p^{th} quantile, distribution function, reliability function and hazard function is given to illustrate application and significant of unified approach of characterization result (2.1) of theorem 2.1.

Example 3.1. Characterization of Generalize Uniform Distribution (GUD) through hazard function

$$g(\theta)=\frac{\left(\frac{\alpha+1}{\theta}\right)\left(\frac{t}{\theta}\right)^{\alpha}}{1-\left(\frac{t}{\theta}\right)^{\alpha+1}}$$

therefore

$$g(X)=\frac{\left(\frac{\alpha+1}{X}\right)\left(\frac{t}{X}\right)^{\alpha}}{1-\left(\frac{t}{X}\right)^{\alpha+1}},$$

From (2.3) one gets $\phi(X)$ as

$$\phi(X)=g(X)+\left(\frac{X}{\alpha+1}\right)\frac{\mathrm{d}}{\mathrm{d}X}g(X)=-\frac{(\alpha+1)t\left(\frac{t}{X}\right)^{2\alpha}}{\left[1-t\left(\frac{t}{X}\right)^{\alpha}\right]^{2}}.$$

and using (2.10) of remark 2.1 the

$$M(X)=\frac{\frac{\mathrm{d}}{\mathrm{d}X}g(X)}{\phi(X)-g(X)}=\frac{1+\alpha}{X}.$$

By characterization method describe in remark 2.1, if

$$\frac{\mathrm{d}}{\mathrm{d}X}\log\left[\frac{X^{a+1}}{a+1}\right]=M(X)$$

then

$$T(X)=\frac{X^{a+1}}{a+1}$$

and substituting $T(X)$ as appeared in (2.12) for (2.11),

$$f(x;\theta)=\frac{\frac{\mathrm{d}}{\mathrm{d}x}T(x)}{T(\theta)}=\frac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha}.$$

is characterized.

Example 3.2. The characterization of $f(x;\theta)$ defined in (2.1) through non constant function such as

$$g_i(\theta)=\begin{cases}\frac{\alpha+1}{\alpha+2}\theta; & \text{for } i=1,\text{ mean,}\\ \frac{\alpha+1}{\alpha+r+1}\theta^{r}; & \text{for } i=2,\ r^{th}\text{ moment,}\\ e^{\theta}; & \text{for } i=3,\\ e^{-\theta}; & \text{for } i=4,\\ \theta p^{1/\alpha+1}; & \text{for } i=5,\ p^{th}\text{ quantile,}\\ \left(\frac{t}{\theta}\right)^{\alpha+1}; & \text{for } i=6,\text{ distribution function,}\\ 1-\left(\frac{t}{\theta}\right)^{\alpha+1}; & \text{for } i=7,\text{ reliability function,}\\ \frac{\left(\frac{\alpha+1}{\theta^{\alpha+1}}\right)t^{\alpha}}{1-\left(\frac{t}{\theta}\right)^{\alpha+1}}; & \text{for } i=8,\text{ hazard function.}\end{cases} \quad (3.1)$$

and using

$$[\phi(X)-g_i(X)]=\begin{cases}\dfrac{X}{\alpha+2}; & \text{for } i=1, \text{ mean},\\ \dfrac{rX^r}{\alpha+r+1}; & \text{for } i=2,\ r^{th} \text{ moment},\\ \dfrac{Xe^X}{\alpha+1}; & \text{for } i=3,\\ -\dfrac{Xe^X}{\alpha+1}; & \text{for } i=4,\\ \dfrac{Xp^{1/\alpha+1}}{\alpha+1}; & \text{for } i=5,\ p^{th} \text{ quantile},\\ -\left(\dfrac{t}{X}\right)^{\alpha+1}; & \text{for } i=6, \text{ distribution function},\\ \left(\dfrac{t}{X}\right)^{\alpha+1}; & \text{for } i=7, \text{ reliability function},\\ -\dfrac{(\alpha+1)\left(\frac{t}{X}\right)^{\alpha}}{X^2\left[\left(\frac{t}{X}\right)^{\alpha+1}-1\right]^2}; & \text{for } i=8, \text{ hazard function}.\end{cases} \tag{3.2}$$

and defining $M(X)$ given in (2.10) and substituting $T(X)$ as appeared in (2.12) for (2.11), $f(x;\theta)$ is characterized.

Example 3.3. In context of remark 2.2 uniform distribution with pdf given in (2.13) characterized through p^{th} quantile

$$g(\theta)=Q_p(\theta)=\theta p$$

therefore

$$g(X)=Xp.$$

From (2.3) one gets $\phi(X)$ as

$$\phi(X)=g(X)+X\frac{\mathrm{d}}{\mathrm{d}X}g(X)=2Xp$$

and using (2.10) of remark 2.1 the

$$M(X)=\frac{\frac{\mathrm{d}}{\mathrm{d}X}g(X)}{\phi(X)-g(X)}=\frac{1}{X}.$$

By characterization method describe in remark 2.1, if

$$\frac{\mathrm{d}}{\mathrm{d}X}\log[X]=\frac{1}{X}=M(X)$$

then

$$T(X)=X$$

$$f(x;\theta)=\frac{\frac{\mathrm{d}}{\mathrm{d}x}T(x)}{T(\theta)}=\frac{1}{\theta}$$

Example 3.3. The pdf $f(x;\theta)$ defined in (2.13) can be characterized through non-constant function such as

$$g_i(\theta)=\begin{cases}\frac{\theta}{2}; & \text{for } i=1, \text{ mean,}\\ \frac{\theta^r}{r+1}; & \text{for } i=2, r^{th} \text{ moment,}\\ e^{\theta}; & \text{for } i=3,\\ e^{-\theta}; & \text{for } i=4,\\ \theta p; & \text{for } i=5, p^{th} \text{ quantile,}\\ \frac{t}{\theta}; & \text{for } i=6, \text{ distribution function,}\\ 1-\frac{t}{\theta}; & \text{for } i=7, \text{ reliability function,}\\ \frac{1}{\theta-t}; & \text{for } i=8, \text{ hazard function.}\end{cases}$$

and using

$$\left[\phi_i(X)-g_i(X)\right]=\begin{cases}\frac{X}{2}; & \text{for } i=1, \text{ mean,}\\ \frac{rX^r}{r+1}; & \text{for } i=2, r^{th} \text{ moment,}\\ Xe^{X}; & \text{for } i=3,\\ -Xe^{X}; & \text{for } i=4,\\ Xp; & \text{for } i=5, p^{th} \text{ quantile,}\\ -\frac{t}{X}; & \text{for } i=6, \text{ distribution function,}\\ \frac{t}{X}; & \text{for } i=7, \text{ reliability function,}\\ -\frac{X}{(X-t)^2}; & \text{for } i=8, \text{ hazard function.}\end{cases}$$

and defining $M(X)$ given in (2.10) and substituting $T(X)$ as appeared in (2.12) for (2.11), $f(x;\theta)$ is characterized.

4. Conclusion

To characterize pdf defined in (1.1) one needs any arbitrary non-constant function of X which should be differentiable and integrable only.

Acknowledgements

This work is supported by UGC Major Research Project No: F.No.42-39/2013 (SR), dated 12-3-2013.

References

[1] Ooms, G. and Moore, K.L. (1991) A Model Assay for Genetic and Environmental Changes in the Architecture of Intact Roots Systems of Plants Grown *in Vitro*. *Plant Cell, Tissue and Organ Culture*, **27**, 129-139. http://dx.doi.org/10.1007/BF00041281

[2] Dixit, U.J., Ali, M. and Woo, J. (2003) Efficient Estimation of Parameters of a Uniform Distribution in the Presence of Outliers. *Soochow Journal of Mathematics*, **29**, 363-369.

[3] Dixit, U.J. and Phal, K.D. (2009) Estimation of P (X Y) for the Uniform Distribution in the Presence of Outliers. *Journal of Probability and Statistical Science*, **7**, 11-18.

[4] Bartoszynski, R. (1980) Personal Communication.

[5] Terrel, G.R. (1983) A Characterization of Rectangular Distributions. *The Annals of Probability*, **11**, 823-826. http://dx.doi.org/10.1214/aop/1176993533

[6] Lopez-Bldzquez, F. and Salamanca Milño, B. (1999) On Terrers Characterization of Uniform Distribution. *Statistical Papers*, **40**, 335-342. http://dx.doi.org/10.1007/BF02929879

[7] Kent, J.T., Mardia, K.V. and Rao, J.S. (1979) Characterization of the Uniform Distribution on the Circle. *The Annals of Statistics*, **7**, 882-889. http://dx.doi.org/10.1214/aos/1176344737

[8] Lin, G.D. (1988) Characterizations of Uniform Distributions and of Exponential Distributions. *Sankhyā A*, **50**, 64-69.

[9] Too, Y.H. and Lin, G.D. (1989) Characterizations of Uniform and Exponential Distributions. *Statistics & Probability Letters*, **7**, 357-359. http://dx.doi.org/10.1016/0167-7152(89)90051-5

[10] Arnold, B.C. and Meeden, G. (1976) A Characterization of the Uniform Distribution Based on Summation Modulo One, with Application to Fractional Backlogs. *Australian Journal of Statistics*, **18**, 173-175. http://dx.doi.org/10.1111/j.1467-842X.1976.tb01294.x

[11] Driscoll, M.F. (1978) On Pairwise and Mutual Independence: Characterizations of Rectangular Distributions. *Journal of the American Statistical Association*, **73**, 432-433. http://dx.doi.org/10.1080/01621459.1978.10481596

[12] Shimizu, R. and Huang, J.S. (1983) On a Characteristic Property of the Uniform Distribution. *Annals of the Institute of Statistical Mathematics*, **35**, 91-94. http://dx.doi.org/10.1007/BF02480966

[13] Abdelhamid, S.N. (1985) On a Characterization of Rectangular Distributions. *Statistics and Probability Letters*, **3**, 235-238. http://dx.doi.org/10.1007/BF02480966

[14] Chow, Y. and Huang, S.Y. (1999) A Characterization of the Uniform Distribution via Moments of N-Fold Convolution modulo One. *The Indian Journal of Statistics*, **61**, 148-151.

[15] Sumitra, P. and Subir, B. (1990) Characterization of Uniform Distributions by Inequalities of Chernoff-Type. *The Indian Journal of Statistics*, **52**, 376-382.

DOI: 10.13189/ms.2014.020602

Characterization of Generalized Uniform Distribution Through Expectation of Function of Order Statistics

Bhatt Milind B.

Department of Statistics, Sardar Patel University, Vallabh Vidyanagar-388120, Anand, Gujarat, India

*Corresponding Author: bhattmilind_b@yahoo.com

Abstract Normally the mass of a root has a uniform distribution but some of have different uniform distribution named generalized uniform distribution (GUD). The characterization result based on expectation of function of order statistics has been obtained for generalized uniform distribution. Applications are given for illustrative purpose.

Keywords Generalize Uniform Distribution, Uniform Distribution, Probability Density Function

1 Introduction

Plant develops into the reproductive phase of growth, a mat of smaller roots grows near the surface to a depth of approximately $(\frac{1}{6})^{th}$ of maximum depth achieve [See G. Ooms and K. L. Moore [1]]. Dixit [2] studied problem of efficient estimation of parameters of a uniform distribution in the presence of outliers. He assumed that a set of random variables $X_1, X_2, , X_n$ represents the masses of roots where out of n random variables some of these roots (say k) have different masses therefore, those masses have different uniform distribution with unknown parameters and these k observations are distributed with Generalize Uniform Distribution (GUD) with probability density function (pdf)

$$f(x;\theta) = \begin{cases} \frac{\alpha+1}{\theta^{\alpha+1}} x^{\alpha}; & a < x < \theta < b; \alpha > -1, a \equiv 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $-\infty < a < b < \infty$ are known constants, x^{α} is positive absolutely continuous function and $(\frac{1}{\theta})^{\alpha+1}$. ince derivative of $x^{\alpha+1}$ and since range is truncated by θ from left $a \equiv 0$. Dixt [3] obtained Maximum Likelihood Estimator (MLE) and the Uniformly Minimum Variance (UMVU) estimator of reliability functional; $P\lfloor X > Y \rfloor$ hat the UMVUE is better than MLE when one parameter of GUD is known, where as both parameters of the GUD are unknown, $P\lfloor X > Y \rfloor$ estimated by using mixture estimate and is consistent.

In this paper characterizing property of GUD with pdf given in (1) has been studied which also holds for uniform distribution on interval $(0, \theta)$ when $\alpha = 0$ in (1). Various approaches used to characterize uniform distribution, few of them have used coefficient of correlation of smaller and the larger of a random sample of size two, Bartoszyn'ski [4], Terreel [5], Lopez -Bldzquez [6] were as Kent [7], has used independence of sample mean and variance, Lin [8], Too [9], Arnold [10], Driscoll [11], Shimizu [12] ,Abdelhamid [13] have used moment conditions, n-fold convolution modulo one and inequalities of chernoff-type also used see Chow [14] and Sumrita [15].

In contrast to all above brief research background and application of characterization of member of Pearson family, this research not provide unified approach to characterize generalize uniform distribution.

The aim of the present research note is to give path breaking new characterization for generalize uniform distribution through expectation of function of order statistics, using identity and equality of expectation. Characterization theorem derived in section 2 with method for characterization as remark and section 3 devoted to applications for illustrative purpose including special case of uniform distribution.

2 Characterization theorem

Page NO. 69

Theorem

Let $X_1, X_2, , X_n$ be a random sample of size n from distribution function F. Let $X_{1:n} < X_{2:n}, ..., < X_{n:n}$ be the set of corresponding order statistics.Assume that F is continuous on the interval (a, b) where $-\infty < a < b < \infty$. Let $g(X_{n:n})$ and $\phi(X_{n:n})$ be be two distinct differentiable and intregrable functions of n^{th} order statistic; $X_{n:n}$ on the interval (a, b) where $-\infty < a < b < \infty$ and moreover $g(X_{n:n})$ be non-constant function of $X_{n:n}$. Then

$$E\Big[g(X_{n:n}) + \Big(\frac{X_{n:n}}{n(\alpha+1)}\Big)\frac{d}{dX_{n:n}}g(X_{n:n})\Big] = g(\theta), \tag{2}$$

is the necessary and sufficient condition for pdf $f(x;\theta)$ of F to be $f(x;\theta)$ defined in (1).

Proof

Given $f(x;\theta)$ defined in (1), for necessity of (2) if $\phi(X_{n:n})$ is such that $g(\theta) = E[\phi(X_{n:n})]$ where $g(\theta)$ is differentiable function then using $f(X_{n:n};\theta)$; pdf of n^{th} order statistic one gets,

$$g(\theta) = \int_a^\theta \phi(x_{n:n}) f(x_{n:n};\theta)\mathrm{d}x_{n:n}. \tag{3}$$

Differentiating (2) with respect to θ on both sides and replacing $X_{n:n}$ for θ, and simplifying one gets

$$\phi(x_{n:n}) = g(x_{n:n}) + \Big(\frac{x_{n:n}}{n(\alpha+1)}\Big)\frac{d}{dx_{n:n}}g(x_{n:n}), \tag{4}$$

which establishes necessity of (2). Conversely given (2), let $k(X_{n:n};\theta)$ be the p.d.f. of pdf of n^{th} order statistic such that

$$g(\theta) = \int_a^\theta \Big[g(x_{n:n}) + \Big(\frac{x_{n:n}}{n(\alpha+1)}\Big)\frac{d}{dx_{n:n}}g(x_{n:n})\Big]k(x_{n:n};\theta)\mathrm{d}x_{n:n}, \tag{5}$$

Since $a = 0$, the following identity holds

$$g(\theta) \equiv \frac{1}{\theta^{n(\alpha+1)}}\int_a^\theta \Big[\frac{d}{dx_{n:n}}g(x_{n:n})x_{n:n}^{n(\alpha+1)}\Big]\mathrm{d}x_{n:n}. \tag{6}$$

Differentiating integrand $g(x_{n:n})x_{n:n}^{n(\alpha+1)}$ with respect to $x_{n:n}$ and simplifying after taking $\frac{d}{dx_{n:n}}x_{n:n}^{n(\alpha+1)}$ as one factor one gets (6)

$$\begin{aligned} g(\theta) \equiv \int_a^\theta &\Big[g(X_{n:n}) + \Big(\frac{x_{n:n}^{n(\alpha+1)}}{\frac{d}{dX_{n:n}}x_{n:n}^{n(\alpha+1)}}\Big)\frac{d}{dX_{n:n}}g(X_{n:n})\Big] \\ &\cdot\Big(\frac{\alpha+1}{\theta^{\alpha+1}}\frac{d}{dx_{n:n}}x_{n:n}^{n(\alpha+1)}\Big)\mathrm{d}x_{n:n}. \end{aligned} \tag{7}$$

Substituting derivative of $x_{n:n}^{n(\alpha+1)}$ (7) one gets (7) as

$$g(\theta) \equiv \int_a^\theta \phi(x_{n:n})\Big(\frac{\alpha+1}{\theta^{\alpha+1}}x_{n:n}^{n(\alpha+1)}\Big)\mathrm{d}x_{n:n}, \tag{8}$$

where $\phi(X_{n:n})$ is as derived in (4). By uniqueness theorem from (5) and (8)

$$k(x_{n:n};\theta) = \frac{\alpha+1}{\theta^{\alpha+1}}x_{n:n}^{n(\alpha+1)}. \tag{9}$$

Since $\frac{1}{a^\alpha}$ is decreasing function for $-\infty < a < b < \infty$ and $a^\alpha = 0$ is satisfy only when range of $x_{n:n}$ is truncated by θ from right and integrating (6) on the interval (a, θ) on both sides, one gets (8) as

$$1 = \int_a^\theta k(x_{n:n};\theta)\mathrm{d}x_{n:n}. \tag{10}$$

For $n = 1$, $[k(x_{n:n};\theta)]_{n=1}$ reduces to $f(x;\theta)$ defined in (1). Hence sufficiency of (2) establishes.

Remark 2.1

Using $\phi(X_{n:n})$ derived in (4), the $f(x;\theta)$ given in (1) can be determined by

$$M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}}g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n}))}. \tag{11}$$

and pdf is given by

Page NO. 70

$$f(x;\theta)=\Big[\frac{\frac{d}{dX_{n:n}}T(X_{n:n})}{T(X_{n:n})}\Big]_{n=1}. \tag{12}$$

where $T(X_{n:n})$ is increasing function for $-\infty<a<b<\infty$ with $T(a)=0$ such that it satisfies

$$M(X_{n:n})=\frac{d}{dX_{n:n}}\log T(X_{n:n}). \tag{13}$$

Examples

Using method describe in remark generalize uniform distribution (GUD) through expectation of non-constant function of order statistics such as mean, rth raw moment, e^{θ} ,$e^{-\theta}$, pth quantile, distribution function, reliability function and hazard function is given to illustrate application and significant of unified approach of characterization result.

Examples 3.1 Characterization of generalize uniform distribution (GUD) through Uniformly Minimum Variance Unbiased Estimator (UMVUE); $\widehat{\lambda}(t)$ of $\lambda(t)$ the hazard function is given to illustrate application and significant of unified approach of characterization result

$$g(X_{n:n})=-\frac{1}{n}\Big(\frac{t^{\alpha}}{t^{\alpha}-X_{n:n}^{\alpha}}\Big)\Big[\frac{\alpha X_{n:n}^{\alpha}}{t^{\alpha}-X_{n:n}^{\alpha}}+n(\alpha+1)\Big]=\widehat{\lambda}(t). \tag{14}$$

Using (4) one gets

$$\begin{aligned}\phi(x_{n:n})&=g(x_{n:n})+\Big(\frac{x_{n:n}}{n(\alpha+1)}\Big)\frac{d}{dx_{n:n}}g(x_{n:n})\\&=-t^{\alpha}\Big(\frac{2\alpha X_{n:n}^{\alpha}+n(\alpha+1)}{n(t^{\alpha}-X_{n:n}^{\alpha})^2}\\&\quad+\frac{\alpha^2X_{n:n}^{\alpha}(t^{\alpha}-X_{n:n}^{\alpha})}{n^2(\alpha+1)(t^{\alpha}-X_{n:n}^{\alpha})^3}\Big)\end{aligned} \tag{15}$$

and (11) of remark 2.1 will be

$$M(X_{n:n})=\frac{\frac{d}{dX_{n:n}}g(X_{n:n})}{\phi(X_{n:n})-g(X_{n:n}))}=\frac{\alpha+1}{X_{n:n}}. \tag{16}$$

By characterizing method given in remark 2.1

$$\frac{d}{dX_{n:n}}\log(X_{n:n}^{n(\alpha+1)})=\frac{\alpha+1}{X_{n:n}}=M(X_{n:n}), \tag{17}$$

then

$$T(X_{n:n})=X_{n:n}^{n(\alpha+1)}, \tag{18}$$

and

$$f(x;\theta)=\Big[\frac{\frac{d}{dx_{n:n}}T(x_{n:n})}{T(x_{n:n})}\Big]_{n=1}=\frac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha}. \tag{19}$$

Examples 3.2 Characterization of generalize uniform distribution (GUD) through Uniformly Minimum Variance Unbiased Estimator (UMVUE); $\widehat{\theta^r}$ of θ^r is given to illustrate application and significant of unified approach of characterization result

$$g(X_{n:n})=\frac{n\alpha+n+r}{n(\alpha+n+r)}X_{n:n}^{r}=\widehat{\theta^r}. \tag{20}$$

Using (4) one gets

$$\begin{aligned}\phi(x_{n:n})&=g(x_{n:n})+\Big(\frac{x_{n:n}}{n(\alpha+1)}\Big)\frac{d}{dx_{n:n}}g(x_{n:n})\\&=\frac{(n\alpha+n+r)^r}{n^2(\alpha+1)(\alpha+r+1)}X_{n:n}^{r}\end{aligned} \tag{21}$$

and repeating (16) to (19), (1) is characterized.

Page NO. 71

Examples 3.3 Using the uniformly minimum variance unbiased (UMVU) estimator $\widehat{g}(\theta)$ and maximum likelihood estimator (MLE) $\widetilde{g}(\theta)$ of $g(\theta)$ such as mean; $\mu_1'(\theta)$, rth moment; $\mu_r'(\theta)$, e^{θ}, $e^{-\theta}$, pth quantile; $Q_p(\theta)$, distribution function; $F(t)$; reliability function; $\bar{F}(t)$, hazard rate; $\lambda(t)$, GUD characterized to illustrate application and significant of unified approach of characterization result. The UMVU estimators

$$\widehat{g}(\theta) = \begin{cases} \widehat{\mu_1'(\theta)} = \frac{n\alpha+n+1}{\alpha+1} X_{n:n}; & \text{for i = 1,} \\ \widehat{\mu_r'(\theta)} = \frac{n\alpha+n+r}{n(\alpha+r+1)} X_{n:n}^r, & \text{for i = 2,} \\ \widehat{e^{\theta}} = \left[1 + \frac{X_{n:n}}{n(\alpha+1)}\right] e^{X_{n:n}}; & \text{for i = 3,} \\ \widehat{e^{-\theta}} = \left[1 - \frac{X_{n:n}}{n(\alpha+1)}\right] e^{-X_{n:n}}; & \text{for i = 4,} \\ \widehat{Q_p}(\theta) = p^{\frac{1}{\alpha+1}} \left[1 + \frac{1}{n(\alpha+1)}\right] X_{n:n}; & \text{for i = 5,} \\ \widehat{F}(t) = \left[1 - \frac{1}{n}\right] \left(\frac{t}{X_{n:n}}\right)^{\alpha+1}; & \text{for i = 6,} \\ \widehat{\bar{F}}(t) = 1 - \left[1 - \frac{1}{n}\right] \left(\frac{t}{X_{n:n}}\right)^{\alpha+1}, & \text{for i = 7,} \\ \widehat{\lambda}(t) = \frac{(\alpha+1)t^{\alpha}}{X_{n:n}^{\alpha+1} - t^{\alpha+1}} \left[1 - \frac{1}{n(X_{n:n}^{\alpha+1} - t^{\alpha+1})}\right] & \text{; for i = 8,} \end{cases} \tag{22}$$

and MLE

$$\widetilde{g}(\theta) = \begin{cases} \widetilde{\mu_1'(\theta)} = \dfrac{\alpha+1}{\alpha+2} X_{n:n}; \\ \qquad \text{for i = 9,} \\ \widetilde{\mu_r'(\theta)} = \dfrac{\alpha+1}{(\alpha+r+1)} X_{n:n}^r; \\ \qquad \text{for i = 10,} \\ \widetilde{e^{\theta}} = e^{X_{n:n}}; \\ \qquad \text{for i = 10,} \\ \widetilde{e^{-\theta}} = e^{-X_{n:n}} \\ \qquad \text{for i = 10,} \\ \widetilde{Q_p}(\theta) = p^{\alpha+1} X_n : n; \\ \qquad \text{for i = 13,} \\ \widetilde{F}(t) = \left(\dfrac{t}{X_{n:n}}\right)^{\alpha+1}; \\ \qquad \text{for i = 14,} \\ \widetilde{\bar{F}}(t) = 1 - \left(\dfrac{t}{X_{n:n}}\right)^{\alpha+1}; \\ \qquad \text{for i = 15,} \\ \widetilde{\lambda}(t) = \dfrac{(\alpha+1)t^{\alpha}}{X_{n:n}^{\alpha+1} - t^{\alpha+1}}, \\ \qquad \text{for i = 16,} \end{cases} \tag{23}$$

one gets

$$[\phi_i(X_{n:n}) - \widehat{g}_i(\theta)] = \begin{cases} \dfrac{n\alpha+n+1}{n^2(\alpha+1)(\alpha+2)} X_{n:n}; & \text{for i = 1,} \\[2ex] \dfrac{r(n\alpha+n+r)}{n^2(\alpha+1)(\alpha+r+1)} X_{n:n}^r; & \text{for i = 2,} \\[2ex] \dfrac{X_{n:n}}{n(\alpha+1)}\Big[1+\dfrac{X_{n:n}}{n(\alpha+1)} \; \dfrac{1}{n(\alpha+1)}\Big]e^{X_{n:n}}; & \text{for i = 3,} \\[2ex] \Big[\Big(\dfrac{X_{n:n}}{n(\alpha+1)}\Big)^2 - \dfrac{X_{n:n}}{n(\alpha+1)} - \dfrac{X_{n:n}}{(n(\alpha+1))^2}\Big]e^{-X_{n:n}}; & \text{for i = 4,} \\[2ex] \dfrac{X_{n:n}P^{\frac{1}{\alpha+1}}}{n(\alpha+1)}\Big[1+\dfrac{1}{n(\alpha+1)}\Big]; & \text{for i = 5,} \\[2ex] -\dfrac{1}{n}\Big[1-\dfrac{1}{n}\Big]\Big(\dfrac{t}{X_{n:n}}\Big)^{\alpha+1}; & \text{for i = 6,} \\[2ex] \dfrac{1}{n}\Big[1-\dfrac{1}{n}\Big]\Big(\dfrac{t}{X_{n:n}}\Big)^{\alpha+1}; & \text{for i = 7;} \\[2ex] A; & \text{for i = 8,} \end{cases} \quad (24)$$

where

$$A = \Big[n^2\Big(t^{\alpha+1} - X_{n:n}^{\alpha+1}\Big)^3\Big]^{-1}\Big[(\alpha+1)t^\alpha X_{n:n}^\alpha \Big((n+1)t^{\alpha+1} - (n-1)X_{n:n}^\alpha\Big)\Big]$$

and

Page NO. 73

$$
[\phi_i(X_{n:n}) - \tilde{g}(\theta)] = \begin{cases} \dfrac{X_{n:n}}{n(\alpha+2)} \text{ ; for i = 9,} \\ \dfrac{rX_{n:n}}{n(\alpha+r+1)} \text{ ; for i = 10,} \\ \dfrac{X_{n:n}}{n(\alpha+1)} e^{X_{n:n}} \text{ ; for i = 11,} \\ -\dfrac{X_{n:n}}{n(\alpha+1)} e^{-X_{n:n}} \text{ ; for i = 12,} \\ \dfrac{X_{n:n} P^{\frac{1}{\alpha+1}}}{n(\alpha+1)} \text{ ; for i = 13,} \\ -\dfrac{1}{n}\left(\dfrac{t}{X_{n:n}}\right)^{\alpha+1} \text{ ; for i = 14,} \\ \dfrac{1}{n}\left(\dfrac{t}{X_{n:n}}\right)^{\alpha+1} \text{ ; for i = 15,} \\ \dfrac{(\alpha+1)t^{\alpha} X_{n:n}^{\alpha+1}}{n(X_{n:n}^{\alpha+1} - t^{\alpha+1})} \text{ ; for i = 16,} \end{cases} \tag{25}
$$

respectively.

Then by defining $M(X_{n:n})$ given in (11) and substituting $T(X_{n:n})$ as appeared in (13) for (12), $f(x;\theta)$ given in (1)is characterized.

Examples 3.4 In context of remark 2.2 uniform distribution on interval $(0, \theta)$ with pdf given in (15) characterized through (UMVU) estimator;

$$
g(X_{n:n}) = \left[1 - \frac{1}{n}\right]\left(\frac{X_{n:n}}{n}\right) = \widehat{Q_p}, \tag{26}
$$

of Q_p the p^{th} quantile is given to illustrate application and significant of unified approach of characterization result given in remark 2.2.

For'$\alpha = 0$, using (4) one gets

$$
\begin{aligned} \phi(X_{n:n}) &= g(X_{n:n}) + \left(\frac{X_{n:n}}{n}\right)\frac{d}{dX_{n:n}} g(X_{n:n}) \\ &= p\left(1 + \frac{1}{n}\right)^2 X_{n:n} \end{aligned} \tag{27}
$$

and for $\alpha = 0$, (11) will be

$$
M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}} g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n}))} = \frac{n}{X_{n:n}}. \tag{28}
$$

By characterizing method given in remark 2.2 for $f(x;\theta)$ given in (15) for $\alpha = 0$, one gets

$$
\frac{d}{dX_{n:n}} \log(X_{n:n}^n) = \frac{n}{X_{n:n}} = M(X_{n:n}), \tag{29}
$$

then

$$
T(X_{n:n}) = X_{n:n}^n, \tag{30}
$$

and

$$
f(x;\theta) = \left[\frac{\frac{d}{dx_{n:n}} T(x_{n:n})}{T(x_{n:n})}\right]_{n=1} = \frac{1}{\theta}. \tag{31}
$$

given in (15) characterized.

Page NO.74

Examples 3.5

Using method given in remark 2.2 for $f(x;\theta)$ given in (15), characterized through (UMVU) estimator $\widehat{g}(\theta)$ and maximum likelihood estimator (MLE) $\widetilde{g}(\theta)$ of $g(\theta)$ such as mean; $\mu_1^{'}(\theta)$, rth moment; $\mu_r^{'}(\theta)$, e^{θ}, $e^{-\theta}$, pth quantile; $Q_p(\theta)$, distribution function; $F(t)$; reliability function; $\bar{F}(t)$, hazard rate; $\lambda(t)$. The UMVU estimators

$$\widehat{g}(\theta) = \begin{cases} \widehat{\mu_1^{'}(\theta)} = \left[1+\frac{1}{n}\right]\frac{X_{n:n}}{2} & ;\text{ for i = 1,} \\ \widehat{\mu_r^{'}(\theta)} = \left[1+\frac{r}{n}\right]\frac{rX_{n:n}^r}{r+1} & ;\text{ for i = 2,} \\ \widehat{e^{\theta}} = \left[1+\frac{X_{n:n}}{n}\right]e^{X_{n:n}} & ;\text{ for i = 3,} \\ \widehat{e^{-\theta}} = \left[1-\frac{X_{n:n}}{n}\right]e^{-X_{n:n}} & ;\text{ for i = 4,} \\ \widehat{Q_p}(\theta) = p\left[1+\frac{1}{n}\right]X_{n:n}; & \text{for i = 5,} \\ \widehat{F}(t) = \left[1-\frac{1}{n}\right]\left(\frac{t}{X_{n:n}}\right) & ;\text{ for i = 6,} \\ \widehat{\bar{F}}(t) = 1-\left[1-\frac{1}{n}\right]\left(\frac{t}{X_{n:n}}\right) & ;\text{ for i = 7,} \\ \widehat{\lambda}(t) = \frac{1}{X_{n:n}-t}\left[1-\frac{1}{n(X_{n:n}-t)}\right] & ;\text{ for i = 8,} \end{cases} \quad (32)$$

and MLE

$$\widetilde{g}(\theta) = \begin{cases} \widetilde{\mu_1^{'}(\theta)} = \frac{X_{n:n}}{2}.\ ;\text{ for i = 9,} \\ \widetilde{\mu_r^{'}(\theta)} = \frac{X_{n:n}^r}{r+1}\ ;\text{ for i = 10,} \\ \widetilde{e^{\theta}} = e^{X_{n:n}}\ ;\text{ for i = 11,} \\ \widetilde{e^{-\theta}} = e^{-X_{n:n}}\ ;\text{ for i = 12,} \\ \widetilde{Q_p}(\theta) = pX_{n:n}\ ;\text{ for i = 13,} \\ \widetilde{F}(t) = \frac{t}{X_{n:n}};\text{for i = 14,} \\ \widetilde{\bar{F}}(t) = 1-\left(\frac{t}{X_{n:n}}\right)\ ;\text{ for i = 15,} \\ \widetilde{\lambda}(t) = \frac{1}{X_{n:n}-t}\ ;\text{ for i = 16,} \end{cases} \quad (33)$$

one gets

Page NO. 75

$$[\phi_i(X_{n:n}) - \widehat{g}_i(\theta)] = \begin{cases} \left[1+\frac{1}{n}\right]\frac{X_{n:n}}{2n} \text{ ; for i = 1,} \\ \left[1+\frac{r}{n}\right]\frac{rX^r_{n:n}}{n(r+1)} \text{ ; for i = 2,} \\ \frac{X_{n:n}}{n}\left[1+\frac{X_{n:n}}{n} + \frac{1}{n}\right]e^{X_{n:n}}; \text{ for i = 3,} \\ \frac{X_{n:n}}{n}\left[\frac{X_{n:n}}{n} - 1 - \frac{1}{n}\right]e^{-X_{n:n}} \text{ ; for i = 4,} \\ \frac{X_{n:n}P.}{n}\left[1+\frac{1}{n.}\right] \text{ ; for i = 5,} \\ -\frac{n-1}{n^2}\left(\frac{t}{X_{n:n}}\right) \text{ ; for i = 6,} \\ \frac{n-1}{n^2}\left(\frac{t}{X_{n:n}}\right) \text{ ; for i = 7,} \\ \left(n^2(t-X_{n:n})^3\right)^{-1}\left[X_{n:n}\left(t+nt + X_{n:n} - nX_{n:n}\right)\right] \text{ ; for i = 8,} \end{cases} \quad (34)$$

and

$$[\phi_i(X_{n:n}) - \widetilde{g}(\theta)] = \begin{cases} \frac{X_{n:n}}{2n} \text{ ; for i = 9,} \\ \frac{rX^r_{n:n}}{n(r+1)} \text{ ; for i = 10,} \\ \frac{X_{n:n}}{n}e^{X_{n:n}} \text{ ; for i = 11,} \\ -\frac{X_{n:n}}{n}e^{-X_{n:n}} \text{ ; for i = 12,} \\ \frac{X_{n:n}}{n}p \text{ ; for i = 13,} \\ -\frac{1}{n}\left(\frac{t}{X_{n:n}}\right) \text{ ; for i = 14,} \\ \frac{1}{n}\left(\frac{t}{X_{n:n}}\right) \text{ ; for i = 15,} \\ -\frac{X_{n:n}}{n(X_{n:n}-t)^2}; \text{ ; for i = 16,} \end{cases} \quad (35)$$

respectively.

Then using characterizing method descibed in remark 2.2, for $\alpha = 0$ define $M(X_{n:n})$ given in (11) and substituting $T(X_{n:n})$ as appeared in (13) for (12), $f(x;\theta)$ is characterized.

Acknowledgements

Page NO. 76

This work was supported by UGC Major Research Project : F.No.42-39/2013(SR) dated 12-3-2013

REFERENCES

[1] G. Ooms, and K. L. Moore. A model assay for genetic and environmental changes in the architecture of intact roots systems of plants grown in vitro, Plant Cell, Tissue and Organ Culture, 27, 129-139, 1991.

[2] U. J. Dixit and Masoom, Ali and J. Woo. Efficient estimation of parameters of a uniform distribution in the presence of outliers, Soochow Journal of mathematics, Volume 29, No. 4, pp. 363-369, October 2003.

[3] U. J. Dixit and K. D. Phal. Estimation of P (X Y) for the uniform distribution in the Presence of Outliers, Journal of Probability and Statistical Science, 7(1), 11-18, Feb, 2009.

[4] R. Bartoszynski. Personal communication, 1980.

[5] G. R. Terrel. A characterization of rectangular distributions. The Annals of Probability, Vol. 11, N 3, 823-826, 1983.

[6] F. Lopez -Bldzquez and B. Salamanca Milno. On Terrers characterization of uniform distribution Statistical Papers 40, 335-342, 1999.

[7] J. T. Kent, K. V. Mardia and J. S. Rao. Characterization of the uniform distribution on the circle. The Annals of Statistics, Vol. 7, No. 4, 882-889, 1979.

[8] G. D. Lin. Characterizations of uniform distributions and of exponential distributions. Sankhya A, 50: 64-69, 1988.

[9] Y. H. Too and G. D. Lin. Characterizations of uniform and exponential distributions. Statistics and Probability Letters 7 357-359, 1989.

[10] B. C. Arnold and G. Meeden. A characterization of the uniform distribution based on summation modulo one, with application to fractional backlogs. Austral. J. Statist., 18, (3), 173-175, 1976.

[11] M. F. Driscoll. On pairwise and mutual independence: characterizations of rectangular distributions, J. Amer. Statist. Assoc. 73, 432-433, 1978.

[12] R. Shimizu and J. S. Huang. On a characteristic property of the uniform distribution. Ann. Inst. Statist. Math., 35, 91-94, 1983.

[13] S. N. Abdelhamid. On a characterization of rectangular distributions. Statistics and Probability Letters, 3, 235-238, 1985.

[14] Y. Chow and S. Huang. A characterization of the uniform distribution via moments of n-fold convolution modulo one. Sankhya : The Indian Journal of Statistics, Volume 61, Series A, Pt. 1, pp. 148-151, 1999.

[15] P. Sumitra. and B. Subir. Characterization of uniform distributions by inequalities of chernoff-type. Sankhya ? : The Indian Journal of Statistics 1990, Volume 52, Series A, Pt. 3, pp. 376-382, 1990.

ProbStat Forum, Volume 08, January 2015, Pages 34–44
ISSN 0974-3235

ProbStat Forum is an e-journal.
For details please visit www.probstat.org.in

Characterization of one-truncation parameter family of distributions through expectation

Bhatt Milind. B

Department of Statistics, Sardar Patel University, Vallabh Vidyanagar - 388120, Anand, Gujarat, India.

Abstract. For characterization of one (left or right)-truncation parameter families of distributions (which includes notably negative exponential distribution, Pareto distribution, power function distribution, uniform distribution and generalized uniform distribution as special case) one needs any arbitrary non-constant function only in place of various approaches such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy available in the literature. Path breaking different approach for characterization of general setup of one-truncation parameter family of distributions through expectation of any arbitrary non constant differentiable function of random variable is obtained. Applications and examples are given for illustrative purpose.

1. Introduction

One-truncation parameter family of distributions with probability density function (pdf)

$$f_j(x;\theta) = \begin{cases} q_j(\theta)h_j(x) & \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $-\infty \le a < b \le \infty$ are known constant, $a < \theta < x < b$ for $j = 1$, $a < x < \theta < b$ for $j = 2$, $h_j; (j = 1, 2)$ are positive absolutely continuous functions, $q_j; (j = 1, 2)$ are everywhere differentiable functions is characterized.

Since $h_j(.);(j = 1 \text{ or } 2)$ is positive and the range is truncated by truncation parameter θ from left or right respectively $q_1^{-1}(b) = q_2^{-1}(a) = 0$.

Using identity of distribution and equality of expectation of function of function, characterization for general set up of one (left or right)-truncation parameter family of distributions defined in (1) through expectation of any arbitrary non-constant differentiable function is given which includes characterization of negative exponential distribution, Pareto distribution as special case of $f_1(x;\theta)$ where as power function distribution, uniform distribution, generalize uniform distribution as special case of $f_2(x;\theta)$.

Keywords. characterization, truncation parameter families of distributions, negative exponential distribution, Pareto distribution, power function distribution, uniform distribution, generalized uniform distribution
Received: 15 October 2013; Accepted: 22 December 2014
Email address: bhattmilind_b@yahoo.com (Bhatt Milind. B)

Several characterizations of these distributions by various approaches are available in the literature. Notably for power function distribution independence of suitable function of order statistics and distributional properties of transformation of exponential variable used by Fisz (1958), Basu (1965), Govindarajulu (1966) and Dallas (1976), linear relation of conditional expectation used by Beg and Kirmani (1974), recurrence relations between expectations of function of order statistics used by Ali and Khan (1998), record valves used by Nagraja (1977), lower record statistics used by Faizan and Khan (2011), product of order statistics used by Arslan (2011) and Lorenz curve used by Moothathu (1986) are available in the literature.

Other approaches such as coefficient of correlation of order statistics of sample of size two used by Bartoszynski (1980), Terrel (1983), Fernando and Rebollo (1997), maximal correlation coefficient between order statistics of identically distributed spacings *etc.* [used by Stapleton (1963), Arnold and Meeden (1976), Driscoll (1978), Shimizu and Huang (1983), Abdelhamid (1985)], moment conditions used by Lin (1988), Too and Lin (1989), moments of n-fold convolution modulo one used by Chow and Huang (1999), inequalities of chernoff-type used by Sumitra and Subir (1990) for characterization of uniform distribution.

Various approaches were used for characterization of negative exponential distribution. Amongst many other Fisz (1958), Tanis (1964), Rogers (1963) and Ferguson (1967) used properties of identical distributions, absolute continuity, constant regression of adjacent order statistics, Ferguson (1964, 1965) and Crawford (1966), used linear regression of adjacent order statistics of random, independent and non degenerate random variables, Nagaraja (1977, 1988) used linear regression of two adjacent record values were as Khan *et al.* (2009) used difference of two conditional expectations, conditioned on a non-adjacent order statistics to characterized negative exponential distribution.

Economic variation in reported income and true income used by Krishnaji (1970), Nagesh *et al.* (1974), independence of suitable function of order statistics used by Malik (1970), Ahsanullah and Kabir (1974), Shah and Kabe (1981) and Dimaki and Xekalaki (1993), mean and the extreme observation of the sample used by Srivastava (1965), linear relation of conditional expectation used by Beg and Kirmani (1974), Dallas (1976), recurrence relations between expectations of function of order statistics used by Ali and Khan (1998), exponential and related distributions used by Tavangar and Asadi (2010), for characterization of Pareto distribution.

Necessary and sufficient conditions for pdf $f(x;\theta)$ to be $f_j(x;\theta)$, (j = 1 or 2), defined in (1) is established in section 2. Section 3 is devoted for applications where as section 4 is devoted to examples for illustrative pùrpose.

2. Characterization theorem

Let X be a random variable (r.v) with distribution function F. Assume that F is continuous on the interval (a, b), where $-\infty \leq a < b \leq \infty$. Let $g(X)$ be a non-constant differentiable function of X on the interval (a, b), where $-\infty \leq a < b \leq \infty$ and more over $g(X)$ be non constant. Then $f(x;\theta)$ is $f_j(x;\theta)$, pdf defined in (1) if and only if

$$E\Big[g(X) + \frac{\frac{d}{dX}g(X)}{M(X)}\Big] = g(\theta) \tag{2}$$

where $M(X)$ is finite function.

Proof. Given $f_j(x;\theta)$ defined in (1), for necessity of (2) if $\phi(X)$ is such that $g(\theta) = E[\phi(X)]$ where $g(\theta)$ is differentiable function then

$$g(\theta) = \begin{cases} \int_\theta^b \phi(x) f_1(x;\theta) \mathrm{d}x for j = 1 \\ \int_a^\theta \phi(x) f_2(x;\theta) \mathrm{d}x for j = 2, \end{cases} \tag{3}$$

Differentiating with respect to θ on both sides of (3) and replacing X for θ, and denoting finite function

$$M(X) = \frac{d}{dX}\left[\log\left(q_j^{-1}(X)\right)\right], \tag{4}$$

and simplifying one gets

$$\phi(X) = g(X) + \frac{\frac{d}{dX}g(X)}{M(X)}, \tag{5}$$

which establishes the necessity of (2). Conversely given (2) let $k_j(x;\theta), j = 1,2$ be pdf of r.v X such that

$$g(\theta) = \begin{cases} \int_\theta^b \phi(x) k_1(x;\theta) \mathrm{d}x for j = 1 \\ \int_a^\theta \phi(x) k_2(x;\theta) \mathrm{d}x for j = 2 \end{cases} \tag{6}$$

Since $q_1^{-1}(b) = q_2^{-1}(a) = 0$ the following identity holds.

$$g(\theta) = \begin{cases} -q_1(\theta) \int_\theta^b \left[\frac{d}{dx} g(x) q_1^{-1}(x)\right] \mathrm{d}x for j = 1 \\ q_2(\theta) \int_\theta^b \left[\frac{d}{dx} g(x) q_2^{-1}(x)\right] \mathrm{d}x for j = 2 \end{cases} \tag{7}$$

Differentiating the integrand of (7) $g(x)q_j^{-1}(x)$, $(j = 1,2)$ and taking $\frac{d}{dx}q_j^{-1}(x)$ as one factor one gets

$$g(\theta) = \begin{cases} \int_\theta^b \phi(x) \left[-q_1(\theta) \frac{d}{dx} q_1^{-1}(x)\right] \mathrm{d}x for j = 1 \\ \int_a^\theta \phi(x) \left[q_2(\theta) \frac{d}{dx} q_2^{-1}(x)\right] \mathrm{d}x for j = 2, \end{cases} \tag{8}$$

where $\phi(X)$ is a function of X derived in (5) for $j = 1,2$. From (6) and (8) by uniqueness theorem

$$k_j(x;\theta) = \begin{cases} -q_1(\theta) \frac{d}{dx} q_1^{-1}(x) for j = 1 \\ q_2(\theta) \frac{d}{dx} q_2^{-1}(x) for j = 2. \end{cases} \tag{9}$$

Since q_1 is increasing function of θ with $q_1^{-1}(b) = 0$ and q_2 is decreasing function of θ with $q_2^{-1}(a) = 0$ integrating (9) on both sides one gets

$$1 = \begin{cases} \int_\theta^b k_1(x;\theta) \mathrm{d}x for j = 1 \\ \int_\theta^b k_2(x;\theta) \mathrm{d}x for j = 2. \end{cases} \tag{10}$$

and denoting

$$h_j(x) = (-1)^j \frac{d}{dx} q_j^{-1}(x), \tag{11}$$

one gets (10) as

$$k_j(x;\theta) = \begin{cases} q_j(\theta) h_j(x) \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

Hence $k_j(x;\theta)$ reduces to $f_j(x;\theta)$ defined in (1) which establishes sufficiency of (2).

Remark. Using $\phi(X)$ given in (5) one can determine $f_j(x;\theta)$ by

$$M(X) = \frac{\frac{d}{dX} g(X)}{\phi(X) - g(X)}, \tag{13}$$

and pdf is given by

$$f_j(x;\theta) = (-1)^j \frac{\frac{d}{dX} q_j^{-1}(x)}{q_j^{-1}(\theta)}, \tag{14}$$

where $q_j^{-1}(x)$ is decreasing function for $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(b) = 0$ for $j = 1$ and $q_j^{-1}(x)$ is increasing function for $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(a) = 0$ for $j = 2$ such that it satisfies (4) for $j = 1, 2$.

3. Special cases, characterizations of various distributions

As special cases of the characterization theorem following distributions are characterized.

(A) Characterization of negative exponential distribution with pdf

$$f_3(x;\theta) = \begin{cases} \exp -(x-\theta); & a < \theta < x < b, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) - \frac{d}{dX} g(X)\Big] = g(\theta) \tag{16}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as -1 and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = -1 \Rightarrow q_j^{-1}(X) = \exp(-X), \tag{17}$$

which is decreasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(b) = 0$ therefore $-\infty \leq a < \theta < x < b \leq \infty$ and by using (11)

$$h_j(x) = \exp(-X). \tag{18}$$

Substituting these values in (1) for j=1, $f_1(x;\theta)$ reduces to $f_3(x;\theta)$ defined in (15). Thus negative exponential distribution is characterized.

(B) Characterization of Pareto distribution with pdf

$$f_4(x;\theta) = \begin{cases} \frac{c\theta^c}{x^{c+1}}; & a < \theta < x < b. \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) - \frac{X}{c}\frac{d}{dX}g(X)\Big] = g(\theta) \tag{20}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as $-c/X$ and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = -\frac{X}{c} \Rightarrow q_j^{-1}(X) = \frac{1}{cX^c}, \tag{21}$$

which is decreasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(b) = 0$ therefore $-\infty \leq a < \theta < x < b \leq \infty$ and by using (11)

$$h_j(X) = \frac{1}{X^{c+1}}. \tag{22}$$

Substituting these values in (1) for j=1, $f_1(x;\theta)$ reduces to $f_4(x;\theta)$ defined in (19). Thus Pareto distribution is characterized.

(C) Characterization of power function distribution with pdf

$$f_5(x;\theta) = \begin{cases} c\theta^{-c}x^{c-1}; & a < x < \theta < b, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) + \frac{X}{c}\frac{d}{dX}g(X)\Big] = g(\theta) \tag{24}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as c/X and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = \frac{X}{c} \Rightarrow q_j^{-1}(X) = \frac{X^c}{c}, \tag{25}$$

which is increasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(a) = 0$ therefore $-\infty \leq a < x < \theta < b \leq \infty$ and by using (11)

$$h_j(X) = X^{c+1}. \tag{26}$$

Substituting these values in (1) for $j = 2$, $f_2(x;\theta)$ reduces to $f_5(x;\theta)$ defined in (23). Thus power function distribution is characterized.

(D) Characterization of uniform distribution with pdf

$$f_6(x;\theta) = \begin{cases} \frac{1}{\theta} & a < x < \theta < b, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) + X\frac{d}{dX}g(X)\Big] = g(\theta) \tag{28}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as $1/X$ and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = \frac{X}{c} \Rightarrow q_j^{-1}(X) = X, \tag{29}$$

which is increasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(a) = 0$ therefore $-\infty \leq a < x < \theta < b \leq \infty$ and by using (11)

$$h_j(x) = 1. \tag{30}$$

Substituting these values in (1) for $j = 2$, $f_2(x;\theta)$ reduces to $f_6(x;\theta)$ defined in (27). Thus uniform distribution is characterized.

(E) Characterization of generalized uniform distribution with pdf

$$f_6(x;\theta) = \begin{cases} \frac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha} & a < x < \theta < b, \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) + \frac{X}{\alpha+1}\frac{d}{dX}g(X)\Big] = g(\theta) \tag{32}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as $\frac{\alpha+1}{X}$ and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = \frac{X}{c} \Rightarrow q_j^{-1}(X) = \frac{X^{\alpha+1}}{\alpha+1}, \tag{33}$$

which is increasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(a) = 0$ therefore $-\infty \leq a < x < \theta < b \leq \infty$ and by using (11)

$$h_j(x) = X^{\alpha}. \tag{34}$$

Substituting these values in (1) for $j = 2$, $f_2(x;\theta)$ reduces to $f_6(x;\theta)$ defined in (31). Thus uniform distribution is characterized.

4. Examples

Let $\phi_i(X) = X$ and $E[X] = g_i(\theta)$, $i = 1, 2, 3, 4, 5$ where

$$g_i(\theta) = \begin{cases} \frac{c}{c+1}\theta; & \text{for i} = 1, \\ \frac{\theta}{2}; & \text{for i} = 2, \\ \frac{\alpha+1}{\alpha+2}; & \text{for i} = 3, \\ \theta + 1; & \text{for i} = 4, \\ \frac{c}{c-1}\theta; & \text{for i} = 5, \end{cases} \tag{35}$$

be means and let

$$\phi_i(X) = \begin{cases} \frac{c+1}{c} X p^{-\frac{1}{c}}; & \text{for i} = 6, \\ 2Xp; & \text{for i} = 7, \\ \frac{\alpha+2}{\alpha+1} X p^{\frac{1}{\alpha+1}}; & \text{for i} = 8, \\ -\log(1-p) + X - 1; & \text{for i} = 9, \\ \frac{c-1}{c} X (1-p)^{-\frac{1}{c}}; & \text{for i} = 10, \\ -\frac{\left(\frac{c}{t}\right)\left(\frac{t}{X}\right)^{2c}}{\left[\left(\frac{t}{X}\right)^{c} - 1\right]^2}; & \text{for i} = 11, \\ -\frac{t}{(t-X)^2}; & \text{for i} = 12, \\ \frac{(\alpha+1)t\left(\frac{t}{X}\right)^{2\alpha}}{\left[X - t\left(\frac{t}{X}\right)^{\alpha}\right]^2}; & \text{for i} = 13, \end{cases} \tag{36}$$

be such that $E[\phi_i(X)] = g_i(\theta)$ where

$$g_i(\theta) = \begin{cases} \theta p^{\frac{1}{c}}; & \text{for i} = 6, \\ \theta p; & \text{for i} = 7, \\ \theta p^{\frac{1}{\alpha+1}}; & \text{for i} = 8, \\ -\log(1-p)+\theta; & \text{for i} = 9, \\ \theta(1-p)^{-\frac{1}{c}}; & \text{for i} = 10, \\ -\frac{\left(\frac{c}{t}\right)}{\left(\frac{\theta}{t}\right)^{c}-1}; & \text{for i} = 11, \\ \frac{1}{(\theta-t)}; & \text{for i} = 12, \\ \frac{\left(\frac{\alpha+1}{\theta}\right)\left(\frac{t}{\theta}\right)^{\alpha}}{1-\left(\frac{t}{\theta}\right)^{\alpha+1}}; & \text{for i} = 13, \end{cases} \tag{37}$$

is pth quantile for $i = 6, 7, 8, 9, 10$ and is hazard function for $i = 11, 12, 13$.

Using (13) we get

$$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X)-g(X)} = \begin{cases} \frac{c}{X}; & \text{for i} = 1,\ 6,\ 11\ , \\ \frac{1}{X}; & \text{for i} = 2,\ 7,\ 12, \\ \frac{\alpha+1}{X}; & \text{for i} = 3,\ 8,\ 13, \\ -1; & \text{for i} = 4,\ 9, \\ -\frac{c}{X}; & \text{for i} = 5,\ 10. \end{cases} \tag{38}$$

Since $q_j^{-1}(X)$ is decreasing function for $-\infty \le a < b \le \infty$ with $q_j^{-1}(b) = 0$ for $j = 1$ and $q_j^{-1}(X)$ is increasing function for $-\infty \le a < b \le \infty$ with $q_j^{-1}(a) = 0$ for $j = 2$, using (4) in (38) it follows that

$$q_j^{-1}(X) = \begin{cases} \frac{X^c}{x}; & \text{for i} = 1,\ 6,\ 11 \text{ and j} = 2, \\ X; & \text{for i} = 2,\ 7,\ 12, \text{ and j} = 2, \\ \frac{X^{\alpha+1}}{\alpha+1}; & \text{for i} = 3,\ 8,\ 13, \text{ and j} = 2, \\ \exp(-X); & \text{for i} = 4,\ 9, \text{ and j} = 1, \\ -\frac{X^{-c}}{c}; & \text{for i} = 5,\ 10 \text{ and j} = 1, \end{cases} \tag{39}$$

and by using (11) one gets

$$h_j(x) = (-1)^j \frac{d}{dX} q_j^{-1}(x) = \begin{cases} x^{c-1}; & \text{for i = 1, 6, 11 and j = 2,} \\ 1; & \text{for i = 2, 7, 12, and j = 2,} \\ x^{\alpha}; & \text{for i = 3, 8, 13, and j = 2,} \\ \exp(-x); & \text{for i = 4, 9, and j = 1,} \\ x^{-c-1}; & \text{for i = 5, 10 and j = 1.} \end{cases} \quad (40)$$

Using method described in the remark the pdf $f_j(x,\theta)$ defined in (1) can be characterized through expectation of a function of random variable; $E[\phi_i(X)] = g_i(\theta)$; $i = 1, 2, ..., 13$ non constant functions by substituting M(X) defined in (13) and using $q_j^{-1}(X)$ as appear in (4) and using (11) for (14) as follows :

j	i	$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X)-g(X)}$	$h_j(X) = (-1)^j \cdot \frac{d}{dX}g(X)$	$q_j^{-1}(X) \ni M(X) = \ni \frac{d}{dX}\log\left(q_j^{-1}(X)\right)$	$f_j(x,\theta) = q_j(\theta)h_j(x) = (-1)^j \frac{\frac{d}{dx}q_j^{-1}(X)}{q_j^{-1}(\theta)}$
1	4, 9	- 1	$\exp(-x)$	$\exp(-x)$	$f_3(x,\theta) = \begin{cases} \exp-(x-\theta); \\ \quad a < \theta < x < b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of negative exponential distribution
1	5, 10	$-\frac{c}{x}$	$\frac{x^{-c}}{c}$	x^{-c-1}	$f_4(x,\theta) = \begin{cases} \frac{c\theta^c}{x^{c+1}}; \\ \quad a < \theta < x < b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of Pareto distribution
2	1, 6, 11	- 1	$\frac{c}{x}$	$\frac{x^c}{c}$	$f_5(x,\theta) = \begin{cases} c\theta^{-c}x^{c-1}; \\ \quad a < x < \theta < b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of power function distribution
2	2, 7, 12	$\frac{1}{x}$	x	1	$f_6(x,\theta) = \begin{cases} \frac{1}{\theta}; \\ \quad a < x < \theta < b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of uniform distribution
2	3, 8, 13	$\frac{\alpha+1}{x}$	$\frac{x^{\alpha+1}}{x^{\alpha+1}}$	$x^{\alpha+1}$	$f_7(x,\theta) = \begin{cases} \frac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha}; \\ \quad a < x < \theta < b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of generalize uniform distribution

Acknowledgement

This work was supported by UGC Major Research Project No : F.No.42-39/2013(SR).

References

[1] Abdelhamid, S. N. (1985), *On a characterization of rectangular distributions, Statistics and Probability Letters*, **3**, p. 235-238.

[2] Ahsanullah, M. and Kabir, A. B. M. (1974), *A characterization of the Pareto distribution, Canadian Journal of Statistics*, **2**, p. 95-98.

[3] Ali, M. A. and Khan, A. H. (1998), *Characterization of Some Types of Distributions, Information and Management Sciences*, *9*, *p. 1-9.*

[4] Arnold, B. C. and Meeden, G. (1976), *A characterization of the uniform distribution based on summation modulo one, with application to fractional backlogs, Austral. J. Statist.*, ***18***, *p. 173-175.*

[5] Arslan, G. (2011), *Characterization based on product of order statistics, math.ST., arXiv**1110.2879v 1**.*

[6] Bartoszynski, R. (1980), *Personal communication.*

[7] Basu, A. P. (1965), *On characterizing the exponential distribution by order statistics, Ann. Inst. Statist. Math.*, **17**, p. *93-96.*

[8] Beg, M. L. and Kirmani, S. N. U. A. (1974), *On a characterization of exponential and related distributions, Austral. J. Statist.*, ***16***, *p. 163-166.*

[9] Chow, Y. and Huang, Su-yun. (1999), *A characterization of the uniform distribution via moments of n-fold convolution modulo one, Sankhya : The Indian Journal of Statistics*, ***61***, *Series A, p. 148-151.*

[10] Crawford, G. B. (1966), *Characterizations of geometric and exponential distributions, Ann. Math. Stat.*, ***37***, *p. 1790-1795.*

[11] Dallas, A. C. (1976), *Characterization Pareto and power distribution, Ann. Ins. Statist. Math*, ***28***, *p. 491-497.*

[12] Dimaki, C. and Evdokia Xekalaki. (1993), *Characterizations of the Pareto distribution based on order statistics, in Stability Problems for Stochastic Models, Lecture Notes in Mathematics Volume **1546**, p. 1-16.*

[13] Driscoll, M. F. (1978), *On pairwise and mutual independence: characterizations of rectangular distributions, J. Amer. Statist. Assoc.*, ***73***, *p. 432-433.*

[14] Faizan, M. and Khan, M. I. (2011), *A Characterization of continuous distributions through lower record statistics, ProbStat Forum*, ***4***, *p. 39-43.*

[15] Ferguson, T. S. (1964), *A characterization of the negative exponential distribution, Ann. Math. Stat.*, ***35***, *p. 1199-1207.*

[16] Ferguson, T. S. (1965), *A characterization of the geometric distribution, Amer. Math. Monthly*, ***72***, *p. 256-260.*

[17] Ferguson, T. S. (1967), *on characterizing distributions by properties Of order statistics, the Indian journal of statistics, Series A*, ***29***, *p. 265-278.*

[18] Fernando Lopez Blaquez and Rebollo Luis Moreno, J. (1997), *On terrer's characterization of uniform distribution, Sankhyā the Indian journal of statistics, Series A*, ***59***, *p. 311-323.*

[19] Fisz, M. (1958), *Characterization of some probability distributions, Skand. Aktuarietidskr*, ***41***, *p. 65-67.*

[20] Govindarajulu, Z. (1966), *Characterization of exponential and power distribution, Skand. Aktuarietidskr*, ***49***, *p. 132-136.*

[21] Khan, A. H., Mohd, F. and Ziaul, H. (2009), *Characterization of probability distributions through order statistics, ProbStat Forum*, ***2***, *p. 132-136.*

[22] Krishnaji, N. (1970), *Characterization of the Pareto distribution through a model of underreported incomes, Econometrica*, ***38***, *p. 251-255.*

[23] Lin, G. D. (1988), *Characterizations of distributions via relationships between two moments of order statistics, Sankhyā Indian journal of statistics, Series A*, ***50***, *p. 73-80.*

[24] Malik, H. J. (1970), *Characterization of the Pareto distribution, Skand. Aktuartidskr.*, ***53***, *p. 115-117.*

[25] Moothathu, T. S. K. (1986), *Characterization of power function distribution through property of Lorenz curve, Sankhyā, Indian Journal of Statistics, 48, p. 262–265.*

[26] Nagaraja, H. N. (1977), *On a characterization based on record values, Australian Journal of Statistics, 19, p. 70-73.*

[27] Nagaraja, H. N. (1988), *Some characterization of continuous distributions based on adjacent order statistics and record values, Sankhyā, the Indian Journal of Statistics, Series A, 50, p. 70-73.*

[28] Nagesh, S. R., Michael, J. H. and Marcello, P. (1974), *A characterization of the Pareto distribution, The Annals of Statistics, 2, p. 599-601.*

[29] Rogers, G. S. (1963), *An alternative proof of the characterization of the density AxB, The American Mathematical Monthly, 70, p. 857-858.*

[30] Shah, S. M. and Kabe, D. G. (1981), *Characterizations of exponential, Pareto, power function, Burr and logistic distributions by order statistics, Biometrical Journal, 23, p. 141-146.*

[31] Shimizu, R. and Huang, J. S. (1983), *On a characteristic property of the uniform distribution, Ann. Inst. Statist. Math., 35, p. 91-94.*

[32] Srivastava, M. S. (1965), *Characterizations of Paretos distributions, Ann. Math. Statist., 36, p. 361-362.*

[33] Stapleton, J. H. (1963), *A characterization of the uniform distribution on a compact topological group, Ann. Math. Statist., 34, p. 319-326.*

[34] Sumitra, P. and Subir, B. (1990), *Characterization of uniform distributions by inequalities of Chernoff-type, Sankhya: the Indian Journal of Statistics, Series A, 52, p. 376-382.*

[35] Tanis, E. (1964), *Linear forms in the order statistics from an exponential distributer, Ann. Math. Stat.,35, p. 270-276.*

[36] Tavangar, M. and Asadi, M. (2010), *some new characterization results on exponential and related distributions, Bulletin of the Iranian Mathematical Society, 36, p. 257-272.*

[37] Terrel, G. R. (1983), *A characterization of rectangular distributions, The Annals of Probability, 11, p. 823-826.*

[38] Too, Y. H. and Lin, G. D. (1989), *Characterizations of uniform and exponential distributions, Statistics and Probability Letters, 7, p. 357-359.*

ProbStat Forum, Volume 10, July 2017, Pages 51–62
ISSN 0974-3235

ProbStat Forum is an e-journal.
For details please visit www.probstat.org.in

Characterization of One-Truncation Parameter Family of Distributions Through Expectation of Function of Order Statistics

Bhatt Milind. B

Department of Statistics, Sardar Patel University, Vallabh Vidyanagar, Anand, Gujarat - 388 120, India.

Abstract. For characterization of one (left or right)-truncation parameter families of distributions one needs any arbitrary non-constant function of order statistics only in place of various alternative approaches available in the literature. Path breaking different approach for characterization of general setup of one-truncation parameter family of distributions through expectation of any arbitrary non constant differentiable function of order statistics is obtained. Applications and examples are given for illustrative purpose.

1. Introduction

One-truncation parameter family of distributions with probability density function (pdf)

$$f_j(x;\theta) = \begin{cases} \begin{cases} q_1(\theta)h_1(x); & \text{for } j=1, a<\theta<x<b \\ 0, & \text{otherwise,} \end{cases} \\ \begin{cases} q_2(\theta)h_2(x); & \text{for } j=2, a<x<\theta<b \\ 0, & \text{otherwise,} \end{cases} \end{cases} \tag{1}$$

where $-\infty \leq a < b \leq \infty$ are known constant, $a < \theta < x < b$ for $j = 1$, $a < x < \theta < b$ for $j = 2$, $h_j; (j = 1, 2)$ are positive absolutely continuous functions, $q_j; (j = 1, 2)$ are everywhere differentiable functions is characterized.

Since $h_j(.); (j = 1 \text{ or } 2)$ is positive and the range is truncated by truncation parameter θ from left or right respectively $q_1^{-1}(b) = q_2^{-1}(a) = 0$. Through out the paper $q_j^{-1}(.)$ is reciprocal of $q_j(.)$.

Most powerful application of characterizations of distribution is to address a fundamental problem of identification of an appropriate model that can describe the real situation which generate the observations.

2010 *Mathematics Subject Classification.* 62E10.
Keywords. truncation parameter families of distributions, negative exponential, Pareto, power function distribution, uniform, generalized uniform.
Received: 08 July 2016; Revised: 19 December 2016, Re-revised: 26 March 2017, Accepted: 20 June 2017

Email address: bhattmilind_b@yahoo.com (Bhatt Milind. B)

For instant 60 observations of random phenomena observed and one group of student fit normal distribution where other group fit log-normal distribution with almost same p-value. This is one of case where characterization results provide navigation tools for correct direction of further study (research). Therefore characterizations of distribution is of general interest to mathematical community, to probabilists and statisticians as well as to researchers and practitioner industrial engineering and operation research and various scientist specializing in natural and behavior science, in particular those who are interested in foundation and application of probabilistic model building. Motivated by such future in this paper, identity of distribution and equality of expectation is used to, characterized (left or right)-truncation parameter family of distributions defined in (1) through expectation of any arbitrary non-constant differentiable function of order statistics which includes characterization of negative exponential distribution, Pareto distribution as special case of $f_1(x;\theta)$ where as power function distribution, uniform distribution, generalize uniform distribution as special case of $f_2(x;\theta)$.

Several characterizations of these distributions by various approaches are available in the literature. Notably for power function distribution independence of suitable function of order statistics and distributional properties of transformation of exponential variable used by Fisz (1958), Basu (1965), Govindarajulu (1966) and Dallas (1976), linear relation of conditional expectation used by Beg and Kirmani(1974), recurrence relations between expectations of function of order statistics used by Alli and Khan (1998), record valves used by Nagraja (1977), lower record statistics used by Faizan and Khan(2011), product of order statistics used by Arslan (2011) and Lorenz curve used by Moothathu (1986) are available in the literature.

Other approaches such as coefficient of correlation of order statistics of sample of size two used by Bartoszyn'ski (1980), Terreel (1983), Fernando and Rebollo (1997), maximal correlation coefficient between order statistics, of identically distributed spacings etc [used by Stapleton (1963), Arnold and Meeden (1976), Driscoll, M.F. (1978), Shimizu and Huang (1983), Abdelhamid (1985)], power contraction of order statistics by Navarro(2008), Random translation, dilation and contraction of order Statistics by Imtiyaz, Shah, Khan and Barakat(2014), moment conditions used by Lin (1988), Too and Lin (1989), moments of n-fold convolution modulo one used by Chow and Huang (1999), inequalities of chernoff-type used by Sumrita and Subir (1990) for characterization of uniform distribution.

Various approaches were used for characterization of negative exponential distribution. Amongst many other Fisz (1958), Tanis (1964), Rogers (1963) and Fergusion (1967) used properties of identical distributions, absolute continuity, constant regression of adjacentt order statistics, Fergusion (1964, 1965) and Crawford (1966), used linear regression of adjacent order statistics of random, independent and non degenerate random variables, Nagaraja (1977, 1988) used linear regression of two adjacent record values were as Khan, Mohd and Ziaul (2009) used difference of two conditional expectations, conditioned on a non-adjacent order statistics to characterized negative exponential distribution.

Economic variation in reported income and true income used by Krishnaji (1970), Nagesh (1974), independence of suitable function of order statistics used by Henrick (1970), Ahsanullah (1973, 1974), Shah (1981) and Dimaki and Evdokia (1993), linear relation of conditional expectation used by Beg and Kirmani (1974), Dallas (1976), recurrence relations between expectations of function of order statistics used by Alli and Khan (1998), exponential and related distributions used by Tavangar and Asadi(2010), for characterization of Pareto distribution.

Necessary and sufficient conditions for pdf $f(x;\theta)$ to be $f_j(x;\theta), (j = 1 or 2)$, defined in (1) is established in section 2. Section 3 is devoted for applications where as section 4 is devoted to examples for illustrative purpose.

2. Characterization

Theorem 2.1. *Let $X_1, X_2, ..., X_n$ be a random sample of size n from distribution function $F_j; j = 1, 2$. Let $X_{1:n} < X_{2:n}, ..., < X_{n:n}$ be the set of corresponding order statistics. Assume that $F_j; j = 1, 2$ continuous on the interval (a, b) where $-\infty < a < b < \infty$. Let $g(.)$ be non-constant differentiable function of t^{th} order statistic ($j = 1, t = 1$ and $j = 2, t = n$) on the interval (a, b) where $-\infty < a < b < \infty$. Then p.d.f. $f_j(x; \theta)$ of F_j to be $f_j(x; \theta), j = 1, 2$ defined in (1) if and only if*

$$g(\theta) = E[\phi_j(X_{t:n})] = E\Big[g(X_{t:n}) + \frac{\frac{d}{dX_{t:n}} g(X_{t:n})}{\frac{d}{dX_{t:n}} \log[q_j^{-1}(X_{t:n})]}\Big]. \tag{2}$$

Proof. Given $f_j(x; \theta), j = 1, 2$ defined in (1), for necessity of (2) if $\phi_j(X_{t:n})$ is such that $g(\theta) = E[\phi_j(X_{t:n})]$ where $g(\theta)$ is differentiable function then using $f_j(x_{t:n}; \theta)$; pdf of t^{th} order statistic $j = 1, t = 1$ and $j = 2, t = n$ one gets,

$$g(\theta) = \begin{cases} \int_\theta^b \phi_1(x_{1:n}) f_1(x_{1:n}; \theta) dx_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta \phi_2(x_{n:n}) f_2(x_{n:n}; \theta) dx_{n:n}, & \text{for } j = 2, t = n \end{cases} \tag{3}$$

Based on $f_j(x : \theta)$ given in (1), substituting $f_j(x_{t:n} : \theta)$ for $j = 1, t = 1$ and $j = 2, t = n$,the (3) will be

$$g(\theta) = \begin{cases} \int_\theta^b \phi_1(x_{1:n}) n q_1^n(\theta) q_1^{-n+1}(x_{1:n}; \theta) h_1(x_{1:n}) dx_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta \phi_2(x_{n:n}) n q_2^n(\theta) q_2^{-n+1}(x_{n:n}; \theta) h_2(x_{n:n}) dx_{n:n}, & \text{for } j = 2, t = n \end{cases} \tag{4}$$

After simplification the (4) will be

$$\frac{g(\theta)}{n q_1^n(\theta)} = \int_\theta^b \phi_1(x_{1:n}) q_1^{-n+1}(x_{1:n}; \theta) h_1(x_{1:n}) dx_{1:n}; \text{for } j = 1, t = 1 \tag{5}$$

and

$$\frac{g(\theta)}{n q_2^n(\theta)} = \int_a^\theta \phi_2(x_{n:n}) q_2^{-n+1}(x_{n:n}; \theta) h_2(x_{n:n}) dx_{n:n}; \text{for } j = 1, t = 1 \tag{6}$$

Differentiating (5)and (6) with respect to θ on both sides and replacing $X_{1:n}$ for θ in (5) and replacing $X_{n:n}$ for θ in (6)and simplifying one gets

$$\phi_j(X_{t:n}) = g(X_{t:n}) + \frac{\frac{d}{dX_{t:n}} g(X_{t:n})}{\frac{d}{dX_{t:n}} \log[q_j^{-1}(X_{t:n})]}, j = 1, t = 1 \text{ and } j = 2, t = n. \tag{7}$$

Note that

$$M_j(X_{t:n}) = \frac{d}{dX_{t:n}} \log[q_j^{-1}(X_{t:n})], j = 1, t = 1 \text{ and } j = 2, t = n, \tag{8}$$

is finite function of $X_{t:n}$. Further $\phi_j(X_{t:n})$ derived in (7) reduces to (2). This establishes necessity of (2). Conversely given (2) $k_j(x_{t:n};\theta)$ be any arbitrary non constant integrable function of t^{th} order statistic, $j = 1, t = 1$ and $j = 2, t = n$ such that

$$g(\theta) = \begin{cases} \int_\theta^b \phi_1(x_{1:n}) k_1(x_{1:n};\theta) dx_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta \phi_2(x_{n:n}) k_2(x_{n:n};\theta) dx_{n:n} & \text{for } j = 2, t = n. \end{cases} \tag{9}$$

Since q_1 is increasing function with $q_1^{-1}(b) = 0$ and q_2 is decreasing function with $q_2^{-1}(a) = 0$ following identity holds.

$$g(\theta) \equiv \begin{cases} \int_\theta^b -q_1^n(\theta) \left[\frac{d}{dx_{1:n}} g(x_{1:n}) q_1^{-n}(x_{1:n})\right] dx_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta q_2^n(\theta) \left[\frac{d}{dx_{n:n}} g(x_{n:n}) q_2^{-n}(x_{n:n}) dx_{n:n}, \right. & \text{for } j = 2, t = n. \end{cases} \tag{10}$$

Differentiating integrand of (10) $q_j^{-n}(x_{t:n}) g(x_{t:n})$ and tacking $\frac{d}{dx_{t:n}} q_1^{-n}(x_{t:n})$ as one factor $j = 1, t = 1$ and $j = 2, t = n$ one gets (10) as

$$g(\theta) = \begin{cases} \int_\theta^b \phi_1(x_{1:n}) \left[- q_1^{-n}(\theta) \frac{d}{dx_{1:n}} q_1^{-n}(x_{1:n})\right] dx_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta \phi_2(x_{n:n}) \left[q_2^{-n}(\theta) \frac{d}{dx_{n:n}} q_2^{-n}(x_{n:n})\right] dx_{n:n} & \text{for } j = 2, t = n. \end{cases} \tag{11}$$

where $\phi_j(x_{t:n})$ is as derived in (7)($j = 1, t = 1$ and $j = 2, t = n$). From (9) and (11) one gets

$$k_j(x_{t:n};\theta) = \begin{cases} \begin{cases} -q_1^n(\theta) \frac{d}{dx_{1:n}} q_1^{-n}(x_{1:n}); & \text{for } j = 1, a < \theta < x < b \\ 0, & \text{otherwise,} \end{cases} \\ \begin{cases} q_2^n(\theta) \frac{d}{dx_{n:n}} q_2^{-n}(x_{n:n}); & \text{for } j = 2, a < x < \theta < b \\ 0, & \text{otherwise,} \end{cases} \end{cases} \tag{12}$$

Since q_1 is increasing function with $q_1^{-1}(b) = 0$ and q_2 is decreasing function with $q_2^{-1}(a) = 0$ integrating both sides of (12) on interval (a, b) for $j = 1, t = 1$ and $j = 2, t = n$ one gets

$$1 = \begin{cases} \int_\theta^b k_1(x_{1:n};\theta)\mathrm{d}x_{1:n}, & \text{for } j = 1, t = 1, \\ \int_a^\theta k_2(x_{n:n};\theta)\mathrm{d}x_{n:n}, & \text{for } j = 2, t = n. \end{cases} \tag{13}$$

Using (12) and (13), $\Big[k_j(x_{t:n};\theta)\Big]_{n=1}$ reduces to $f_j(x;\theta)$ defined in (1) which establishes sufficiency of (2). □

Remark : Using $\phi(X_{t:n})$ derived in (7), the $f_j(x;\theta)$ given in (1) can be determined by

$$M_j(x_{t:n}) = \frac{\frac{d}{dX_{t:n}} g(X_{t:n})}{\phi(X_{t:n}) - g(X_{t:n})}. \tag{14}$$

and pdf is given by

$$f_j(x;\theta) = \Big[(-1)^j \frac{\frac{d}{dx_{t:n}} U_j(x_{t:n})}{U_j(\theta)}\Big]_{n=1}; j = 1, 2, \tag{15}$$

where $U_j(X_{t:n})$ is decreasing function for $-\infty \le a < b \le \infty$ with $U(b) = 0$, range must be truncated by truncation parameter θ from left for $j = 1, t = 1$ and is increasing function for $-\infty \le a < b \le \infty$ with $U(a) = 0$, range must be truncated by truncation parameter θ from right for $j = 2, t = n$ such that it satisfies

$$M_j(X_{t:n}) = \frac{d}{dX_{t:n}}\Big(\log(U_j(X_{t:n}))\Big). \tag{16}$$

3. Applications

As special cases of the theorem 2.1 the following distributions are characterized.

(A) Characterization of negative exponential distribution with pdf

$$f_3(x;\theta) = \begin{cases} e^{-(x-\theta)}; & a < \theta < x < b, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{1:n}) - \Big(\frac{1}{n}\Big)\frac{d}{dX_{1:n}} g(X_{1:n})\Big], \tag{18}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 1, t = 1$ $M_1(X_{1:n})$ turns out as $-n$ and hence using (14) and (16)

$$M_1(X_{1:n}) = \frac{d}{dX_{1:n}} \log(U_1(X_{1:n})) = -n \Rightarrow U_1(X_{1:n}) = e^{-nX_{1:n}},$$

which is decreasing function on interval (a, b) with $U_1(b) = 0$ and range must be truncated by truncation parameter θ from left. Substituting these values in (15), $f_1(x;\theta)$ reduces to $f_3(x;\theta)$ defined in (17). Thus negative exponential distribution is characterized.

(B) Characterization of Pareto distribution with pdf

$$f_4(x;\theta) = \begin{cases} \frac{c\theta^c}{x^{c+1}}; & a < \theta < x < b, \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{1:n}) - \Big(\frac{X_{1:n}}{cn}\Big)\frac{d}{dX_{1:n}} g(X_{1:n})\Big], \tag{20}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 1, t = 1$ $M_1(X_{1:n})$ turns out as $-\frac{cn}{X_{1:n}}$ and hence using (14) and (16)

$$M_1(X_{1:n}) = \frac{d}{dX_{1:n}} \log(U_1(X_{1:n})) = -\frac{cn}{X_{1:n}} \Rightarrow U_1(X_{1:n}) = -\frac{1}{cnX_{1:n}^{cn}},$$

which is decreasing function on interval (a, b) with $U_1(b) = 0$ and range must be truncated by truncation parameter θ from left. Substituting these values in (15), $f_1(x;\theta)$ reduces to $f_4(x;\theta)$ defined in (19). Thus Pareto distribution is characterized.

(C) Characterization of power function distribution with pdf

$$f_5(x;\theta) = \begin{cases} c\theta^{-c}x^{c-1}; & a < x < \theta < b, c > 0 \\ 0, & \text{otherwise,} \end{cases} \tag{21}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{n:n}) + \Big(\frac{X_{n:n}}{cn}\Big)\frac{d}{dX_{n:n}} g(X_{n:n})\Big], \tag{22}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 2, t = n$ $M_2(X_{n:n})$ turns out as $\frac{cn}{X_{n:n}}$ and hence using (14) and (16)

$M_2(X_{n:n}) = \frac{d}{dX_{n:n}} \log(U_2(X_{n:n})) = \frac{cn}{X_{n:n}} \Rightarrow U_2(X_{n:n}) = \left(\frac{X^c_{n:n}}{c}\right)^n,$

which is increasing function on interval (a, b) with $U_2(a) = 0$ and range must be truncated by truncation parameter θ from right. Substituting these values in (15), $f_2(x;\theta)$ reduces to $f_5(x;\theta)$ defined in (21). Thus power function distribution is characterized.

(D) Characterization of uniform distribution with pdf

$$f_6(x;\theta) = \begin{cases} \frac{1}{\theta}; & a < x < \theta < b, c > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{23}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{n:n}) + \Big(\frac{X_{n:n}}{n}\Big)\frac{d}{dX_{n:n}} g(X_{n:n})\Big], \tag{24}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 2, t = n$ $M_2(X_{n:n})$ turns out as $\frac{n}{X_{n:n}}$ and hence using (14) and (16)

$M_2(X_{n:n}) = \frac{d}{dX_{n:n}} \log(U_2(X_{n:n})) = \frac{cn}{X_{n:n}} \Rightarrow U_2(X_{n:n}) = X^n_{n:n},$

which is increasing function on interval (a, b) with $U_2(a) = 0$ and range must be truncated by truncation parameter θ from right. Substituting these values in (15), $f_2(x;\theta)$ reduces to $f_6(x;\theta)$ defined in (23). Thus uniform distribution is characterized.

(E) Characterization of generalized uniform distribution with pdf

$$f_7(x;\theta) = \begin{cases} \frac{\alpha+1}{\theta^{\alpha+1}} x^\alpha; & a < x < \theta < b, \alpha > -1, \\ 0, & \text{otherwise}, \end{cases} \tag{25}$$

The sufficient condition in theorem 2.1 being

$$g(\theta) = E\Big[g(X_{n:n}) + \Big(\frac{X_{n:n}}{n(\alpha+1)}\Big)\frac{d}{dX_{n:n}} g(X_{n:n})\Big], \tag{26}$$

where $g(\theta)$ is non-constant function. From (14) for $j = 2, t = n$ $M_2(X_{n:n})$ turns out as $\frac{n(\alpha+1)}{X_{n:n}}$ and hence using (14) and (16)

$M_2(X_{n:n}) = \frac{d}{dX_{n:n}} \log(U_2(X_{n:n})) = \frac{cn}{X_{n:n}} \Rightarrow U_2(X_{n:n}) = X^{n^{n(\alpha+1)}}_{n:n},$

which is increasing function on interval (a, b) with $U_2(a) = 0$ and range must be truncated by truncation parameter θ from right. Substituting these values in (15), $f_2(x;\theta)$ reduces to $f_7(x;\theta)$ defined in (25). Thus uniform distribution is characterized.

4. Example

Example 4.1 Let $g_i(X_{n:n})$ be the uniformly minimum variance unbiased (UMVU) estimator;

$$g_i(X_{1:n}) = \begin{cases} X_{1:n} + 1 - \frac{1}{n}; & \text{for } i = 1, \\ \frac{c}{c-1}[1 - \frac{1}{cn}]X_{1:n}, & \text{for } i = 2, \end{cases} \tag{27}$$

$$g_i(X_{n:n}) = \begin{cases} \frac{X_{n:n}}{c+1}[c + \frac{1}{n}]; & \text{for } i = 3, \\ \frac{X_{n:n}}{2}[1 - \frac{1}{n}]; & \text{for } i = 4, \\ \frac{n\alpha+n+1}{n(\alpha+2)} X_{n:n}; & \text{for } i = 5, \end{cases} \tag{28}$$

of $\mu_1^{'}(\theta) = E(X)$; the first row moment and let the UMVU estimator of p^{th} quantile be

$$g_i(X_{1:n}) = \begin{cases} -\log(1-p) + X_{1:n} - \frac{1}{n}; & \text{for } i = 6, \\ X_{1:n}(1-p)^{-\frac{1}{c}}[1 - \frac{1}{cn}], & \text{for } i = 7, \end{cases} \tag{29}$$

$$g_i(X_{n:n}) = \begin{cases} \left(1 + \frac{1}{cn}\right)p^{-\frac{1}{c}}X_{n:n}; & \text{for } i = 3, \\ \left(1 + \frac{1}{n}\right)pX_{n:n}; & \text{for } i = 4, \\ \left(1 + \frac{1}{n(\alpha+1)}\right)p^{\frac{1}{\alpha+1}}X_{n:n}; & \text{for } i = 5, \end{cases} \tag{30}$$

and let the UMVU estimator of hazard function be

$$g_i(X_{n:n}) = \begin{cases} \left(1 - \frac{1}{n}\right)\left(\frac{t}{X_{n:n}}\right)^c; & \text{for } i = 3, \\ \left(\frac{1}{X_{n:n}-t}\right)\left(1 - \frac{X_{n:n}}{n(X_{n:n}-t)}\right); & \text{for } i = 4, \\ \frac{1}{n}\left(\frac{t^\alpha}{t^\alpha - X_{n:n}^\alpha \alpha}\right)\left[\frac{t^\alpha}{t^\alpha - X_{n:n}^\alpha \alpha} - n(1+\alpha)\right]; & \text{for } i = 5, \end{cases} \tag{31}$$

Using (14) we get $M_j(X_{t:n})$, $(j = 1, t = 1$ and $j = 2, t = n)$

$$M_1(X_{1:n}) = \frac{\frac{d}{dX_{1:n}} g(X_{1:n})}{\phi(X_{1:n}) - g(X_{1:n})} = \begin{cases} -n; & \text{for } i = 1, 6, \\ -\frac{cn}{X_{1:n}}, & \text{for } i = 2, 7, \end{cases} \tag{32}$$

which satisfies

$$M_1(X_{1:n}) = \frac{d}{dX_{1:n}} \Big[\log U_1(X_{1:n}) \Big] \Rightarrow \tag{33}$$

$$U_1(X_{1:n}) = \begin{cases} e^{-nX_{1:n}}; & \text{for } i = 1, 6, \\ \frac{1}{cnX_{1:n}^2}, & \text{for } i = 2, 7, \end{cases} \tag{34}$$

and

$$M_2(X_{n:n}) = \frac{\frac{d}{dX_{n:n}} g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n})} \begin{cases} \frac{cn}{X_{n:n}}; & \text{for } i = 3, 8, 11, \\ \frac{n}{X_{n:n}}; & \text{for } 4, 9, 12, \\ \frac{n(\alpha+1)}{X_{n:n}}; & \text{for } 5, 10, 13, \end{cases} \tag{35}$$

which satisfies

$$M_2(X_{n:n}) = \frac{d}{dX_{n:n}} \Big[\log U_2(X_{n:n}) \Big] \Rightarrow \tag{36}$$

$$U_2(X_{n:n}) = \begin{cases} \left(\frac{X_{n:n}}{c}\right)^c; & \text{for } i = 3, 8, 11, \\ X_{n:n}^n; & \text{for } 4, 9, 12, \\ X_{n:n}^{n(\alpha+1)}; & \text{for } 5, 10, 13, \end{cases} \tag{37}$$

Since $U_1(X_{1:n})$ decreasing function on $-\infty < a < b < \infty$ with $U_1(b) = 0$ and since $U_2(X_{n:n})$ increasing function on $-\infty < a < b < \infty$ with $U_2(a) = 0$ using method described in the remark 2.1 the pd $f_j(X; \theta)$

defined in (1) can be characterized through expectation of function of order statistics $g_i(X_{t:n})$; $t = 1$ or $t = n$ for $i = 1, 2, ..., 13$ the UMVU estimator of non constant function such as first row moment, pth quantile and hazard function by substituting $M_j(X_{t:n})$; $j = 1$ and $t = 1$ or $j = 2$ and $t = n$ defined in (14) and using $U_j(X_{t:n})$; ($j = 1, t = 1$ and $j = 2, t = n$) as appeared in (16) for (15) given below :

j	i	$M_j(X_{t:n}) = \frac{\frac{d}{dX_{t:n}} g(X_{t:n})}{\phi_i(X_{t:n}) - g_i(X_{t:n})}$	$U_j(X_{t:n}) \ni M_j(X_{t:n}) = \frac{d\left(\log(U(X_{t:n}))\right)}{dX_{t:n}}$	$f_j(x,\theta) = (-1^j)\left[\frac{\frac{d}{dX_{t:n}} U_j(X_{t:n})}{U_j(\theta)}\right]_{n=1}$
1	1, 6	$-n$	$e^{-nX_{1:n}}$	$f_3(x;\theta) = \begin{cases} e^{-(x-\theta)}; & a < \theta < x < b, \\ 0, & \text{otherwise}, \end{cases}$
1	2, 7	$-\frac{cn}{X_{1:n}}$	$\frac{1}{cnX_{1:n}^{cn}}$	$f_4(x;\theta) = \begin{cases} \frac{c\theta^c}{x^{c+1}}; & a < \theta < x < b, \\ 0, & \text{otherwise}, \end{cases}$
2	3, 8, 11	$\frac{cn}{X_{n:n}}$	$\left(\frac{X_{n:n}^c}{c}\right)^n$	$f_5(x;\theta) = \begin{cases} c\theta^{-c}x^{c-1}; & a < x < \theta < b, \theta = K^{-1}, K > 0, c > 0, \\ 0, & \text{otherwise}, \end{cases}$
2	4, 9, 12	$\frac{n}{X_{n:n}}$	$X_{n:n}^n$	$f_6(x;\theta) = \begin{cases} \frac{1}{\theta}; & a < x < \theta < b, \\ 0, & \text{otherwise}, \end{cases}$
2	5, 10, 13	$\frac{n(\alpha+1)}{X_{n:n}}$	$X_{n:n}^{n(\alpha+1)}$	$f_7(x;\theta) = \begin{cases} \frac{\alpha+1}{\theta^{\alpha+1}} x^{\alpha}; & a << x < \theta < b, \alpha > -1, \\ 0, & \text{otherwise}, \end{cases}$

References

[1] Abdelhamid, S. N.(1985). On a characterization of rectangular distributions *Statistics and Probability Letters*, *3*, pages 235-238.

[2] Ahsanullah, M. and Kabir, A. B. M. (1974). A characterization of the Pareto distribution. *Canadian Journal of Statistics, Section D, Student corner*, *2*, No. 1 , pages 95-98.

[3] Ali, M. A. and Khan, A. H. (1998). Characterization of Some Types of Distributions. *Information and Management Sciences*, *9*, No. 2, pages 1-9.

[4] Arnold, B. C. and Meeden, G. (1976). A characterization of the uniform distribution based on summation modulo one, with application to fractional backlogs. *Austral. J. Statist.*, *18*, pages 173-175.

Page NO. 99

[5] Arslan, G. (2011). Characterization based on product of order statistics. *math.ST.*, **arXiv 1110.2879v**.

[6] Bartoszynski, R.(1980). Personal communication.

[7] Basu, A. P. (1965). On characterizing the exponential distribution by order statistics. *Ann. Inst. Statist. Math.*, ***17***, pages 93-96.

[8] Beg, M. L. and Kirmani, S. N. U. A. (1974). On characterizing the exponential distribution by order statistics. *Austral. J. Statist.*, ***16***, 163–166.

[9] Chow, Y. and Huang, Su-yun (1999). A characterization of the uniform distribution via moments of n-fold convolution modulo one. *Sankhyā Series A, The Indian Journal of Statistics*, ***61***, Pt. 1, pages 148-151.

[10] Crawford, G. B. (1966). Characterizations of geometric and exponential distributions. *Ann. Math. Stat.*, ***37***, *No. 6*, pages 1790-1795.

[11] Dallas, A. C. (1976). Characterization Pareto and power distribution. *Ann. Ins. Statist. Math.*, ***28***, *Part A*, 491–497.

[12] Dimaki, C. and Evdokia Xekalaki (1993). Characterizations of the Pareto distribution based on order statistics. *Stability Problems for Stochastic Models, Lecture Notes in Mathematics*, ***1546***, pages 1-16.

[13] Driscoll, M. F. (1978). On pairwise and mutual independence: characterizations of rectangular distributions. *J. Amer. Statist. Assoc.*, ***73***, pages 432-433.

[14] Faizan, M. and Khan, M. I. (2011). A Characterization of Continuous Distributions through Lower Record Statistics. *ProbStat Forum*, ***4***, pages 39-43.

[15] Ferguson, T. S. (1964). A characterization of the exponential distribution. *Ann. Math. Stat.*, ***35***, pages 1199-1207.

[16] Ferguson, T. S. (1965). A characterization of the geometric distribution. *Amer. Math. Monthly*, ***72***, pages 256-260.

[17] Ferguson, T. S. (1967). on characterizing distributions by properties Of order statistics. *Sankhyā Series A Indian journal of statistics*, ***29***, pages 265-278.

[18] Fernando Lopez BLaquez and Rebollo Luis Moreno, J. (1997). On terrer's characterization of uniform distribution. *Sankhyā Series A Indian journal of statistics*, ***59***, *Part 3*, pages 311-323.

[19] Fisz, M. (1958). Characterization of some probability distribution. *Skand. Aktuarietidskr*, ***41***, pages 65-67.

[20] Govindarajulu, Z. (1966). Characterization of exponential and power distribution. *Skand, Aktuarietidskr*, ***49***, pages 132-136.

lation, dilation and contraction of order Statistics. Statist. Probab. Lett. 92: 209214.

[21] Imtiyaz, A, Khan, A.H. and Barakat, H. M. (2014). Random translation, dilation and contraction of order Statistics. *Probab. Lett.*, ***92***, 2014, pages 209-214.

[22] Khan, A. H, Mohd, F. and Ziaul, H. (2009). Characterization of Probability Distributions Through Order Statistics. *ProbStat Forum*, ***2***, October 2009, pages 132-136.

[23] Krishnaji, N. (1970). Characterization of the Pareto Distribution Through a Model of Underreported Incomes. *Econometrica*, ***38***, Issue 2, pages 251-255.

[24] Lin, G. D. (1988). Characterizations of distributions via relationships between two moments of order statistics *J. Statist. Plan. Infer.*, ***19***, pages 73-80.

[25] Malik, H. J. (1970). Characterization of the Pareto Distribution. *Skand. Aktuartidskr.*, ***53***, pages 115-117.

[26] Moothathu, T. S. K. (1986). Characterization of power function distribution through property of Lorenz curve. *Sankhyā, Indian Journal of Statistics, Volume* ***48***, 262–265.

[27] Nagaraja, H. N. (1977). On a characterization based on record values. *Australian Journal of Statistics*, ***19***, pages 70-73.

[28] Nagaraja, H. N. (1988). Some characterization of continuous distributions based on adjacent order statistics and record values. *Sankhyā, Series A (1961-2002), Indian Journal of Statistics*, ***50***, *No. 1*, pages 70-73.

[29] Nagesh, S. R, Michael, J. H. and Marcello, P. (1974). A Characterization of the Pareto Distribution. *The Annals of*

Statistics, *2*, No 3, pages 599 -601.

[30] Rogers, G. S. (1963). An alternative proof of the characterization of the density AxB. *The American Mathematical Monthly, 70, No. 8,* pages 857-858.

[31] Shah, S. M. and Kabe, D. G. (1981). Characterizations of Exponential, Pareto, Power Function BURR and Logistic Distributions by Order Statistics. *Biometrical Journal, 23,* Issue 2, pages 141-146.

[32] Shimizu, R. and Huang, J. S. (1983). On a characteristic property of the uniform distribution. *Ann. Inst. Statist. Math., 35,* pages 91-94.

[33] Stapleton, J. H. (1963). A characterization of the uniform distribution on a compact topological group. *Ann. Math. Statist., 34,* pages 319-326.

[34] Sumitra, P. and Subir Kumar, B. (1990). Characterization of uniform distributions by inequalities of chernoff-type. *Sankhyā : The Indian Journal of Statistics., 52, Series APt.3* pages 376-382.

[35] Tanis, E. A. (1964). Linear forms in the order statistics from an exponential distributer. *Ann. Math. Stat., 35, No. 1* pages 270-276.

[36] Tavangar, M. and Asadi, M. (2010). Some new characterization results on exponential and related distributions. *Bulletin of the Iranian Mathematical Society,* **36**, No. 1, pages 257-272.

[37] Terrel, G. R. (1983). A characterization of rectangular distributions. *The Annals of Probability, 11,* No. 3, pages 823-826.

[38] Too, Y. H. and Lin, G. D. (1989). Characterizations of uniform and exponential distributions. *Statistics and Probability Letters, 7,* pages 357-359.